THE CLASSIFICATION OF ORBITS ON CERTAIN EXCEPTIONAL JORDAN ALGEBRA UNDER THE AUTOMORPHISM GROUP.

AKIHIRO NISHIO

Abstract. Let $\mathcal{J}^1$ be the real form of complex simple Jordan algebra with the automorphism group G of type $\mathrm{F}_{4(-20)}$. Explicitly, we give the orbit decomposition of $\mathcal{J}^1$ under the action of G and determine the Lie group structure of stabilizer for each G-orbit on $\mathcal{J}^1$.

0. Introduction.

Let G be an exceptional linear Lie group of type F_4 defined by the automorphism group of an exceptional Jordan algebra. The objective of this article is for $G = \mathrm{F}_{4(-20)}$, to solve the following problem:

A classification of G-orbits:

(A) the decomposition of the space of elements in which G is represented, into equivalence classes or "orbits".

(B) the determining the Lie group structure of the stabilizer for each G-orbit.

The orbit decompositions are given for $G = \mathrm{F}_4^{\mathbb{C}}$ and $\mathrm{F}_{4(4)}$ in [12].

The definition of an exceptional Jordan algebra $\mathcal{J}^1$ with identity E is given in §1. For $X, Y \in \mathcal{J}^1$, the Jordan product of X and Y is denoted by $X \circ Y$. $\mathcal{J}^1$ has the trace $\mathrm{tr}(X)$ as usual, and $(X|Y) = \mathrm{tr}(X \circ Y)$ gives a non-degenerate indefinite inner product. Moreover, $\mathcal{J}^1$ has the cross product of H. Freudenthal $X \times Y$. Then the determinant $\det(X)$, the characteristic polynomial $\Phi_X(\lambda)$ degree 3 and the characteristic roots of X are defined. The linear Lie group $\mathrm{F}_{4(-20)}$ is defined to be the automorphism group of $\mathcal{J}^1$ with the Jordan product. The action of $\mathrm{F}_{4(-20)}$ preserves the identity element E, the trace, the inner product, the cross product, the determinant, the characteristic polynomial and the characteristic roots with multiplicity.

We give the elements E_1, E_2, E_3, P^+, P^-, $Q^+(1) \in \mathcal{J}^1$ in §1 and determine typical orbits in §5. These orbits are the *exceptional hyperbolic planes* $\mathcal{H}(\mathbf{O})$, $\mathcal{H}'(\mathbf{O})$, and the *exceptional null cones* $\mathcal{N}_1^+(\mathbf{O})$, $\mathcal{N}_1^-(\mathbf{O})$, $\mathcal{N}_2(\mathbf{O})$.

Date: 2012/4/21.

2010 *Mathematics Subject Classification.* Primary 20G41, Secondary 17C30, 57S20.

Proposition 0.1. *The following equations hold.*

(0.1.a) $$\mathcal{H}(\mathbf{O}) = Orb_{\mathrm{F}_{4(-20)}}(E_1).$$

(0.1.b) $$\mathcal{H}'(\mathbf{O}) = Orb_{\mathrm{F}_{4(-20)}}(E_2) = Orb_{\mathrm{F}_{4(-20)}}(E_3).$$

(0.1.c) $$\mathcal{N}_1^+(\mathbf{O}) = Orb_{\mathrm{F}_{4(-20)}}(P^+).$$

(0.1.d) $$\mathcal{N}_1^-(\mathbf{O}) = Orb_{\mathrm{F}_{4(-20)}}(P^-).$$

(0.1.e) $$\mathcal{N}_2(\mathbf{O}) = Orb_{\mathrm{F}_{4(-20)}}(Q^+(1)).$$

Let $X \in \mathcal{J}^1$. In §1, we define the minimal space V_X which is the minimal dimensional linear subspace of $\mathcal{J}^1$ being closed under the cross product and containing the elements $E, X \in \mathcal{J}^1$. In order to present the intersection of V_X and typical orbits, we define the elements $E_{X,\lambda_1},\ W_{X,\lambda_1} \in V_X$ with $\lambda_1 \in \mathbb{R}$ in §1 and the traceless component of X: $p(X) \in V_X$. Using the set of all characteristic roots of X with multiplicities and the intersection of V_X and typical orbits, the $\mathrm{F}_{4(-20)}$-orbit of X can be described.

Main Theorem 1. $\mathrm{F}_{4(-20)}$*-orbits on* $\mathcal{J}^1$ *are given as follows.*

(1) *Assume that* $X \in \mathcal{J}^1$ *admits characteristic roots* $\lambda_1 > \lambda_2 > \lambda_3$. *Then there exists a unique* $i \in \{1, 2, 3\}$ *such that*

(i) $\mathcal{H}(\mathbf{O}) \cap V_X = \{E_{X,\lambda_i}\}$,

(ii) $\mathcal{H}'(\mathbf{O}) \cap V_X = \{E_{X,\lambda_{i+1}},\ E_{X,\lambda_{i+2}}\}$ *with* $E_{X,\lambda_{i+1}} \neq E_{X,\lambda_{i+2}}$

where indexes i, $i+1$, $i+2$ *are counted modulo* 3. *In this case,* X *can be transformed to one of the following canonical forms by the action of* $\mathrm{F}_{4(-20)}$.

Cases	*Canonical forms of* X
1. $E_{X,\lambda_1} \in \mathcal{H}(\mathbf{O})$	$\mathrm{diag}(\lambda_1, \lambda_2, \lambda_3)$
2. $E_{X,\lambda_2} \in \mathcal{H}(\mathbf{O})$	$\mathrm{diag}(\lambda_2, \lambda_3, \lambda_1)$
3. $E_{X,\lambda_3} \in \mathcal{H}(\mathbf{O})$	$\mathrm{diag}(\lambda_3, \lambda_1, \lambda_2)$

(2) *Assume that* $X \in \mathcal{J}^1$ *admits characteristic roots* $\lambda_1 \in \mathbb{R}$, $p \pm \sqrt{-1}q$ *with* $p \in \mathbb{R}$ *and* $q > 0$. *Then* X *can be transformed to the following canonical form by the action of* $\mathrm{F}_{4(-20)}$.

	The canonical form of X
4.	$\mathrm{diag}(p, p, \lambda_1) + F_3^1(q)$

(3) *Assume that* $X \in \mathcal{J}^1$ *admits characteristic roots* λ_1 *of multiplicity* 1 *and* λ_2 *of multiplicity* 2. *Then*

(i) $E_{X,\lambda_1} \in \mathcal{H}(\mathbf{O}) \coprod \mathcal{H}'(\mathbf{O})$, (ii) $W_{X,\lambda_1} \in \{0\} \coprod \mathcal{N}_1^+(\mathbf{O}) \coprod \mathcal{N}_1^-(\mathbf{O})$,

(iii) $E_{X,\lambda_1} \in \mathcal{H}(\mathbf{O}) \Rightarrow W_{X,\lambda_1} = 0$, (iv) $W_{X,\lambda_1} \neq 0 \Rightarrow E_{X,\lambda_1} \in \mathcal{H}'(\mathbf{O})$

In this case, X *can be transformed to one of the following canonical forms by the action of* $\mathrm{F}_{4(-20)}$.

Cases	*Canonical forms of X*
5. $E_{X,\lambda_1} \in \mathcal{H}(\mathbf{O})$	$\mathrm{diag}(\lambda_1, \lambda_2, \lambda_2)$
6. $E_{X,\lambda_1} \in \mathcal{H}'(\mathbf{O}),\ W_{X,\lambda_1} = 0$	$\mathrm{diag}(\lambda_2, \lambda_2, \lambda_1)$
7. $W_{X,\lambda_1} \in \mathcal{N}_1^+(\mathbf{O})$	$\mathrm{diag}(\lambda_2, \lambda_2, \lambda_1) + P^+$
8. $W_{X,\lambda_1} \in \mathcal{N}_1^-(\mathbf{O})$	$\mathrm{diag}(\lambda_2, \lambda_2, \lambda_1) + P^-$

(4) *Assume that* $X \in \mathcal{J}^1$ *admits a characteristic root of multiplicity* 3. *Then*

$$p(X) \in \{0\} \coprod \mathcal{N}_1^+(\mathbf{O}) \coprod \mathcal{N}_1^-(\mathbf{O}) \coprod \mathcal{N}_2(\mathbf{O}).$$

In this case, X *can be transformed to one of the following canonical forms by the action of* $\mathrm{F}_{4(-20)}$.

Cases	*Canonical forms of X*
9. $p(X) = 0$	$3^{-1}\mathrm{tr}(X)E$
10. $p(X) \in \mathcal{N}_1^+(\mathbf{O})$	$3^{-1}\mathrm{tr}(X)E + P^+$
11. $p(X) \in \mathcal{N}_1^-(\mathbf{O})$	$3^{-1}\mathrm{tr}(X)E + P^-$
12. $p(X) \in \mathcal{N}_2(\mathbf{O})$	$3^{-1}\mathrm{tr}(X)E + Q^+(1)$

(5) *Under the action of* $\mathrm{F}_{4(-20)}$, *these canonical forms* 1 – 12 *in* (1) – (4) *cannot be transformed from each other.*

The Heisenberg group $\mathrm{H}_{\mathrm{Im}\mathbf{O},\mathbf{O}}$ in the sense of J.A. Wolf [20, 21] and its subgroups $\mathrm{Im}\mathbf{O}$ and $\mathrm{H}_{\mathrm{Im}\mathbf{O},\mathrm{Im}\mathbf{O}}$ are given in §8. The Lie group structure of stabilizer for each $\mathrm{F}_{4(-20)}$-orbit on $\mathcal{J}^1$ is given as follows.

Main Theorem 2. (Orbit types of $\mathrm{F}_{4(-20)}$-orbits on $\mathcal{J}^1$).
The Lie group types of stabilizers of the canonical forms 1–12 *in Main-Theorem 1 are given in the following table.*

Canonical forms	*Types of stabilizers*
1. $\mathrm{diag}(\lambda_1, \lambda_2, \lambda_3)$	$\mathrm{Spin}(8)$
2. $\mathrm{diag}(\lambda_2, \lambda_3, \lambda_1)$	$\mathrm{Spin}(8)$
3. $\mathrm{diag}(\lambda_3, \lambda_1, \lambda_2)$	$\mathrm{Spin}(8)$
4. $\mathrm{diag}(p, p, \lambda_1) + F_3^1(q)$	$\mathrm{Spin}^0(7, 1)$
5. $\mathrm{diag}(\lambda_1, \lambda_2, \lambda_2)$	$\mathrm{Spin}(9)$
6. $\mathrm{diag}(\lambda_2, \lambda_2, \lambda_1)$	$\mathrm{Spin}^0(8, 1)$
7. $\mathrm{diag}(\lambda_2, \lambda_2, \lambda_1) + P^+$	$\mathrm{Spin}(7) \ltimes \mathrm{Im}\mathbf{O}$
8. $\mathrm{diag}(\lambda_2, \lambda_2, \lambda_1) + P^-$	$\mathrm{Spin}(7) \ltimes \mathrm{Im}\mathbf{O}$
9. $3^{-1}\mathrm{tr}(X)E$	$\mathrm{F}_{4(-20)}$
10. $3^{-1}\mathrm{tr}(X)E + P^+$	$\mathrm{Spin}(7) \ltimes \mathrm{H}_{\mathrm{Im}\mathbf{O},\mathbf{O}}$
11. $3^{-1}\mathrm{tr}(X)E + P^-$	$\mathrm{Spin}(7) \ltimes \mathrm{H}_{\mathrm{Im}\mathbf{O},\mathbf{O}}$
12. $3^{-1}\mathrm{tr}(X)E + Q^+(1)$	$\mathrm{G}_2 \ltimes \mathrm{H}_{\mathrm{Im}\mathbf{O},\mathrm{Im}\mathbf{O}}$

1. Preliminaries.

Denote the Cartesian n-power of a set X by $X^n := X \times \cdots \times X$ (n times) such as $\mathrm{SO}(8)^3 = \mathrm{SO}(8) \times \mathrm{SO}(8) \times \mathrm{SO}(8)$. Let $\mathbb{R}$ be the field of real numbers and $\mathbb{C} := \mathbb{R} \oplus \sqrt{-1}\mathbb{R}$ the field of complex numbers. Denote $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. Let V be a $\mathbb{F}$-linear space, $\mathrm{GL}_{\mathbb{F}}(V)$ the group of $\mathbb{F}$-linear automorphism of V, and $\mathrm{End}_{\mathbb{F}}(V)$ the linear space of $\mathbb{F}$-linear endomorphisms on V. For a mapping $f : V \to V$ and $c \in \mathbb{F}$, put $V_{f,c} := \{v \in V|\ f(v) = cv\}$ and $V_f := V_{f,1}$. A subset C in V is said to be a *cone* if $x \in C$ and $\lambda > 0$ imply that $\lambda x \in C$. The *exponential* of $f \in \mathrm{End}_{\mathbb{F}}(V)$ is defined by $\exp f = \sum_{k=0}^{\infty} \frac{f^k}{k!} \in \mathrm{GL}_{\mathbb{F}}(V)$. Let G be a subgroup of $\mathrm{GL}_{\mathbb{F}}(V)$ and ϕ an automorphism on G. Denote the subgroup $\{g \in G|\ \phi g = g\}$ of G as G^{ϕ}. For $v_1, \cdots, v_n \in V$, the pointwize stabilizer of $\{v_1, \cdots, v_n\}$ in G is denoted by $G_{v_1,\cdots,v_n}$. For $v \in V$, the G-*orbit* of v is denoted by $Orb_G(v) := \{gv|\ g \in G\}$.

Let V be an $\mathbb{R}$-linear space. Its complexification $V \otimes_{\mathbb{R}} \mathbb{C}$ denoted by $V^{\mathbb{C}}$. For $f \in \mathrm{End}_{\mathbb{R}}(V)$, its complexification is written by the same letter f. The complex conjugation on $V^{\mathbb{C}}$ with respect to V is denoted by τ: $\tau(u + \sqrt{-1}v) := u - \sqrt{-1}v$ for all $u + \sqrt{-1}v \in V^{\mathbb{C}}$ with $u, v \in V$.

Let V be a $\mathbb{F}$-linear space. A *quadratic form* on V is a mapping $\mathrm{q} : V \to \mathbb{F}$ such that (i) $\mathrm{q}(\lambda v) = \lambda^2 \mathrm{q}(v)$ for all $\lambda \in \mathbb{F}$ and $v \in V$, (ii) the associated symmetric form $\mathrm{q} : V \times V \to \mathbb{F}$: $\mathrm{q}(v, w) := 2^{-1}(\mathrm{q}(v) + \mathrm{q}(w) - \mathrm{q}(v - w))$ is bilinear. The pair (V, q) is called a *quadratic space*. For a quadratic space (V', q') and a $\mathbb{F}-$linear map $f : V \to V'$, the quadratic form f^*q' on V is given by $f^*q'(x) := q'(fx)$. An *isomorphism* $f : (V, \mathrm{q}) \to (V', \mathrm{q}')$ is defined as $f : V \to V'$ is a $\mathbb{F}-$linear isomorphism and $f^*q' = q$. Denote the *orthogonal group* of (V, q) by $\mathrm{O}(V, \mathrm{q}) := \{g \in \mathrm{GL}_{\mathbb{F}}(V)|\ g^*\mathrm{q} = \mathrm{q}\} = \{g \in \mathrm{GL}_{\mathbb{F}}(V)|\ \mathrm{q}(gv, gw) = \mathrm{q}(v, w)\}$ and the *special orthogonal group* of (V, q) by $\mathrm{SO}(V, \mathrm{q}) := \{g \in \mathrm{O}(V, \mathrm{q})|\ \det(g) = 1\}$ where $\det(g)$ is the determinant of $g \in \mathrm{End}_{\mathbb{F}}(V)$. Let k, l be non-negative integers with $k + l > 0$. A quadratic from $\mathrm{q}_{k,l}$ on $\mathbb{R}^{k+l}$ is defined by $\mathrm{q}_{k,l}(x) := -\sum_{i=1}^{k} x_i^2 + \sum_{i=1}^{l} x_{k+i}^2$ for $x = (x_1, \cdots, x_{k+l})$, and denote the quadratic space by $(\mathbb{R}^{k,l}, \mathrm{q}_{k,l})$. Assume that (V, q) is an $\mathbb{R}-$quadratic space and the quadratic form q is not the trivial quadratic from $\mathrm{q}' \equiv 0$. Put the subspace $\mathrm{rad}(V, \mathrm{q}) := \{v \in V|\ \mathrm{q}(v, w) = 0$ for all $w \in V\}$ in V. Then there exist a subspace W of V such that $V = W \oplus \mathrm{rad}(V, \mathrm{q})$ and (W, q) is isomorphic to $(\mathbb{R}^{k,l}, \mathrm{q}_{k,l})$ for some integers k, l. The pair of integers (k, l) depends only on the quadratic form q and is called the *signature* of the quadratic form q.

Let $\mathbf{O}$ be the $\mathbb{R}$-algebra of *octonions* [4, 1, 19] with a base 1, e_1, e_2, e_3, e_4, e_5, e_6, e_7 and the multiplications among them are given as follows: 1 is the unit of $\mathbb{R}$; $e_i^2 = -1$; $e_ie_j + e_je_i = 0$ for $i \neq j$; $e_le_m = e_n$, $e_me_n = e_l$ and $e_ne_l = e_m$ for each $(l, m, n) \in \{(1, 2, 3),\ (3, 5, 6),\ (6, 7, 1),\ (1, 4, 5),\ (3, 4, 7),\ (6, 4, 2),\ (2, 5, 7)\}$. We write e_0 for the unit 1 of $\mathbf{O}$. Let $\mathbf{O}^{\mathbb{C}}$

be the complexification of $\mathbf{O}$ with the complex conjugation τ. Denote $\tilde{\mathbf{O}} := \mathbf{O}$ or $\mathbf{O}^{\mathbb{C}}$. Let $x = \sum_{i=0}^{7} x_i e_i$ and $y = \sum_{i=0}^{7} y_i e_i \in \tilde{\mathbf{O}}$ with $x_i, y_i \in \mathbb{F}$. The *conjugation* is defined by $\overline{x} := x_0 - \sum_{i=1}^{7} x_i e_i$, the *inner product* $(x|y) := \sum_{i=0}^{7} x_i y_i$, the *quadratic form* $\mathrm{n}(x) := (x|x)$, the *vector part* $\mathrm{Im}(x) := 2^{-1}(x - \overline{x})$ and the *scalar part* $\mathrm{Re}(x) := 2^{-1}(x + \overline{x}) = (1|x)$, respectively.

Lemma 1.1. (cf. [4], [1], [17]). *Let $x, y, z, a, b \in \tilde{\mathbf{O}}$.*

(1.1.a) $\quad (xy|xy) = (x|x)(y|y).$

(1.1.b) $\quad (ax|ay) = (a|a)(x|y) = (xa|ya).$

(1.1.d) $\quad (ax|by) + (bx|ay) = 2(a|b)(x|y).$

(1.1.e) $\quad (ax|y) = (x|\overline{a}y), \ (xa|y) = (x|y\overline{a}).$

(1.1.f) $\quad \overline{\overline{x}} = x, \ \overline{x+y} = \overline{x} + \overline{y}, \ \overline{xy} = \overline{y}\,\overline{x}.$

(1.1.g) $$\begin{cases} (x|y) = (y|x) = 2^{-1}(x\overline{y} + y\overline{x}) = 2^{-1}(\overline{x}y + \overline{y}x), \\ x\overline{x} = \overline{x}x = (x|x). \end{cases}$$

(1.1.h) $$\begin{cases} a(\overline{a}x) = (a\overline{a})x, a(x\overline{a}) = (ax)\overline{a}, x(a\overline{a}) = (xa)\overline{a}, \\ a(ax) = (aa)x, a(xa) = (ax)a, x(aa) = (xa)a. \end{cases}$$

(1.1.i) $\quad \overline{b}(ax) + \overline{a}(bx) = 2(a|b)x = (xa)\overline{b} + (xb)\overline{a}.$

(1.1.j) $$\begin{cases} (ax)y + x(ya) = a(xy) + (xy)a, \\ (xa)y + (xy)a = x(ay) + x(ya), \\ (ax)y + (xa)y = a(xy) + x(ay). \end{cases}$$

(1.1.k) $\quad (ax)(ya) = a(xy)a \quad$ (*Moufang's formula*).

(1.1.l) $\quad \mathrm{Re}(xy) = \mathrm{Re}(yx), \ \mathrm{Re}(x(yz)) = \mathrm{Re}(y(zx)) = \mathrm{Re}(z(xy)).$

Denote $\mathrm{Im}\tilde{\mathbf{O}} := \{x \in \tilde{\mathbf{O}} |\ \mathrm{Re}(x) = 0\} = \{x \in \tilde{\mathbf{O}} |\ \overline{x} = -x\}$. The quadratic spaces $(\mathbf{O}, \mathrm{n})$ and $(\mathrm{Im}\mathbf{O}, \mathrm{n})$ are isomorphic to $(\mathbb{R}^{0,8}, \mathrm{q}_{0,8})$ and $(\mathbb{R}^{0,7}, \mathrm{q}_{0,7})$, respectively.

Let $\mathbf{K}$ be a $\mathbb{F}$-subalgebra of $\tilde{\mathbf{O}}$ such that $\mathbf{K}$ have the unit 1 and $\overline{x} \in \mathbf{K}$ for all $x \in \mathbf{K}$. Let $M(n, \mathbf{K})$ be the set of all $n \times n$ matrices with entries in $\mathbf{K}$. For $A \in M(n, \mathbf{K})$ with the (i, j)-entry $a_{ij} \in \mathbf{K}$, let ${}^tA \in M(n, \mathbf{K})$ be the transposed matrix having the (i, j)-entry a_{ji}, $\bar{A} \in M(n, \mathbf{K})$ the conjugate matrix having the (i, j)-entry $\overline{a}_{ij}$, and denote $A^* := {}^t\bar{A} \in M(n, \mathbf{K})$. The matrix in $M(n, \mathbf{K})$ with 1 at the (i, j)-th place and zeros elsewhere is denoted by $E_{i,j}$, and the diagonal matrix $\sum_{i=1}^{n} a_{ii} E_{i,i}$ by $\mathrm{diag}(a_{11}, \cdots, a_{nn})$. In particular, denote $E := \mathrm{diag}(1, \cdots, 1)$ and $I_p := -\sum_{i=1}^{p} E_{i,i} + \sum_{i=1}^{q} E_{p+i,p+i} \in M(p+q, \mathbf{K})$. The subalgebra $\mathbf{C} := \{x_0 + x_1 e_1 |\ x_i \in \mathbb{R}\}$ of $\mathbf{O}$ is isomorphic with the field of complex numbers. We use the following notations about some of classical Lie groups: $\mathrm{O}(n) := \{A \in M(n, \mathbb{R}) |\ {}^tAA = E\}$, $\mathrm{SO}(n) := \{A \in M(n, \mathbb{R}) |\ {}^tAA = E, \ \det(A) = 1\}$, $\mathrm{SU}(n) := \{A \in M(n, \mathbf{C}) |\ A^*A = E, \ \det(A) = 1\}$, $\mathrm{O}(p, q) := \{A \in M(n, \mathbb{R}) |\ {}^tAI_pA =$

$I_p\}$ where $\det(A)$ and $\mathrm{tr}(A)$ denote the determinant of $A \in M(n, \mathbf{C})$ and the trace of $A \in M(n, \mathbf{C})$, respectively. For differential manifolds X and Y, $X \simeq Y$ denotes that X and Y are diffeomorphic. Let G be a Lie group. Its Lie algebra is denoted by $Lie(G)$ and the identity connected component of G by G^0. For Lie groups G and G', $G \cong G'$ denotes that G and G' are isomorphic as Lie group. Let N be a normal subgroup of G and H a subgroup of G. If $G = HN$ and $N \cap H = \{1\}$ hold where 1 denotes the identity element of G, then G is called a *semidirect product of* N *and* H, and is denoted by $H \ltimes N$.

Let $i \in \{1, 2, 3\}$ and indexes $i, i+1, i+2$ be counted modulo 3. For $\xi = (\xi_1, \xi_2, \xi_3) \in \mathbb{C}^3$ and $x = (x_1, x_2, x_3) \in (\mathbf{O}^{\mathbb{C}})^3$, denote the hermitian matrix

$$h(\xi; x) := \begin{pmatrix} \xi_1 & x_3 & \overline{x_2} \\ \overline{x_3} & \xi_2 & x_1 \\ x_2 & \overline{x_1} & \xi_3 \end{pmatrix}.$$

The complex exceptional Jordan algebra $\mathcal{J}^{\mathbb{C}}$ is defined by

$$\mathcal{J}^{\mathbb{C}} := \{X \in M(3, \mathbf{O}^{\mathbb{C}}) |\ X^* = X\} = \{h(\xi; x) |\ \xi \in \mathbb{C}^3, x \in (\mathbf{O}^{\mathbb{C}})^3\}$$

with the Jordan product

$$X \circ Y := 2^{-1}(XY + YX) \quad \text{for } X, Y \in \mathcal{J}^{\mathbb{C}}.$$

Then E is the identity element of the Jordan product. Denote the elements $E_i,\ F_i(x) \in \mathcal{J}^{\mathbb{C}}$ as

$$E_i := E_{i,i}, \quad F_i(x) := xE_{i+1,i+2} + \overline{x}E_{i+2,i+1}$$

where $x \in \mathbf{O}^{\mathbb{C}}$. Then

$$h(\xi_1, \xi_2, \xi_3; x_1, x_2, x_3) = \sum\nolimits_{i=1}^{3}(\xi_i E_i + F_i(x_i)).$$

Let $X = \sum_{i=1}^{3}(\xi_i E_i + F_i(x_i)),\ Y \in \mathcal{J}^{\mathbb{C}}$. The *trace* $\mathrm{tr}(X)$ is defined by $\mathrm{tr}(X) := \xi_1 + \xi_2 + \xi_3$ and the inner product $(X|Y)$ by $(X|Y) := \mathrm{tr}(X \circ Y)$. Then the inner product $(X|Y)$ is a non-degenerate inner product. The *cross product* of H. Freudenthal is defined by

$$X \times Y := 2^{-1}\big(2X \circ Y - \mathrm{tr}(X)Y - \mathrm{tr}(Y)X + (\mathrm{tr}(X)\mathrm{tr}(Y) - (X|Y))E\big)$$

[5] (cf. [8, p.232,(47)], [19], [13]) with $X^{\times 2} := X \times X$. The trilinear from $(X|Y|Z)$ and the *determinant* $\det(X)$ are defined by

$$(X|Y|Z) := (X|Y \times Z), \quad \det(X) := 3^{-1}(X|X|X)$$

respectively. The *characteristic polynomial* $\Phi_X(\lambda)$ of $X \in \mathcal{J}^{\mathbb{C}}$ is defined by $\Phi_X(\lambda) := \det(\lambda E - X)$ and a solution of $\Phi_X(\lambda) = 0$ in $\mathbb{C}$ is called a *characteristic root* of X. From direct calculations, we have the following two lemmas.

Lemma 1.2. (cf. [12]). *Let $i \in \{1,2,3\}$ and indexes $i, i+1, i+2$ be counted modulo 3. Let $X = \sum_{i=1}^{3}(\xi_i E_i + F_i(x_i))$, $Y = \sum_{i=1}^{3}(\eta_i E_i + F_i(y_i)) \in \mathcal{J}^{\mathbb{C}}$. Then the following equations hold:*

$$(X|Y) = \sum{}_{i=1}^{3}\big(\xi_i\eta_i + 2(x_i|y_i)\big), \tag{1.2.a}$$

$$\begin{aligned} X \times Y = &\sum{}_{i=1}^{3} \left(2^{-1}(\xi_{i+1}\eta_{i+2} + \eta_{i+1}\xi_{i+2}) - (x_i|y_i)\right) E_i \\ &+ \sum{}_{i=1}^{3} F_i \left(2^{-1}(\overline{x_{i+1}y_{i+2}} + \overline{y_{i+1}x_{i+2}} - \xi_i y_i - \eta_i x_i)\right), \end{aligned} \tag{1.2.b}$$

$$\det(X) = \xi_1\xi_2\xi_3 + 2\mathrm{Re}(x_1x_2x_3) - \sum{}_{i=1}^{3}\xi_i(x_i|x_i). \tag{1.2.c}$$

Lemma 1.3. (cf. [12]). *Let $X, Y \in \mathcal{J}^{\mathbb{C}}$. Then*

$$\mathrm{tr}(X \times Y) = 2^{-1}(\mathrm{tr}(X)\mathrm{tr}(Y) - (X|Y)), \tag{1.3.a}$$

$$\begin{cases} \text{(i)} & E \times E = E, \quad \text{(ii)}\ X \times E = 2^{-1}(\mathrm{tr}(X)E - X), \\ \text{(iii)} & (X^{\times 2}) \times E = 2^{-1}(\mathrm{tr}(X^{\times 2})E - X^{\times 2}), \\ \text{(iv)} & (X^{\times 2})^{\times 2} = \det(X)X, \\ \text{(v)} & (X^{\times 2}) \times X = 2^{-1}\left(-\mathrm{tr}(X)X^{\times 2} - \mathrm{tr}(X^{\times 2})X \right. \\ & \qquad \left. +(\mathrm{tr}(X^{\times 2})\mathrm{tr}(X) - \det(X))E\right), \end{cases} \tag{1.3.b}$$

$$\begin{aligned} \Phi_X(\lambda) &= \lambda^3 - \mathrm{tr}(X)\lambda^2 + \mathrm{tr}(X^{\times 2})\lambda - \det(X) \\ &= \lambda^3 - \mathrm{tr}(X)\lambda^2 + 2^{-1}(\mathrm{tr}(X)^2 - (X|X))\lambda - \det(X). \end{aligned} \tag{1.3.c}$$

The linear Lie group $\mathrm{F}_4^{\mathbb{C}}$ is defined by

$$\mathrm{F}_4^{\mathbb{C}} := \{g \in \mathrm{GL}_{\mathbb{C}}(\mathcal{J}^{\mathbb{C}})|\ g(X \circ Y) = gX \circ gY\}.$$

The following result is proved in [16], [12] after O. Shukuzawa and I. Yokota [14].

Proposition 1.4. *Let $X, Y, Z \in \mathcal{J}^{\mathbb{C}}$.*
(1) *For all $g \in \mathrm{F}_4^{\mathbb{C}}$,*

$$\mathrm{tr}(gX) = \mathrm{tr}(X). \tag{1.4.a}$$

(2) *The following equations hold.*

$$\begin{aligned} \mathrm{F}_4^{\mathbb{C}} &= \{g \in \mathrm{GL}_{\mathbb{C}}(\mathcal{J}^{\mathbb{C}})|\ \det(gX) = X,\ gE = E\} \\ &= \{g \in \mathrm{GL}_{\mathbb{C}}(\mathcal{J}^{\mathbb{C}})\ |\ \Phi_{gX}(\lambda) = \Phi_X(\lambda)\} \\ &= \{g \in \mathrm{GL}_{\mathbb{C}}(\mathcal{J}^{\mathbb{C}})|\ \det(gX) = X,\ (gX|gY) = (X|Y)\} \\ &= \left\{ g \in \mathrm{GL}_{\mathbb{C}}(\mathcal{J}^{\mathbb{C}}) \ \middle|\ \begin{array}{rcl} (gX|gY|gZ) &=& (X|Y|Z), \\ (gX|gY) &=& (X|Y) \end{array} \right\} \\ &= \{g \in \mathrm{GL}_{\mathbb{C}}(\mathcal{J}^{\mathbb{C}})|\ g(X \times Y) = gX \times gY\}. \end{aligned} \tag{1.4.b}$$

The $\mathbb{R}$-exceptional Jordan algebra $\mathcal{J}$ is defined by

$$\mathcal{J} := \{X \in M(3, \mathbf{O}) |\ X^* = X\} = \{h(\xi; x) |\ \xi \in \mathbb{R}^3,\ x \in \mathbf{O}^3\}$$

with the Jordan product $X \circ Y = 2^{-1}(XY + YX)$. Denote the complex conjugation τ with respect to $\mathcal{J}$ in $\mathcal{J}^{\mathbb{C}}$. We define $\sigma \in \mathrm{GL}_{\mathbb{C}}(\mathcal{J}^{\mathbb{C}})$ by

$$\sigma h(\xi_1, \xi_2, \xi_3; x_1, x_2, x_3) := h(\xi_1, \xi_2, \xi_3; x_1, -x_2, -x_3).$$

Because of $\det(\sigma X) = X$ and $\sigma E = E$, applying (1.4.b), we see $\sigma \in \mathrm{F}_4^{\mathbb{C}}$ and clearly $\sigma^2 = 1$ where 1 denotes the identity element of $\mathrm{F}_4^{\mathbb{C}}$. We consider the complex conjugation $\tau\sigma$ in $\mathcal{J}^{\mathbb{C}}$ and define the involutive automorphism $\widetilde{\tau\sigma}$ of $\mathrm{F}_4^{\mathbb{C}}$ by

$$\widetilde{\tau\sigma}(g) = \tau\sigma g \sigma\tau \quad \text{for } g \in \mathrm{F}_4^{\mathbb{C}}.$$

For $\xi = (\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3$ and $x = (x_1, x_2, x_3) \in \mathbf{O}^3$, denote

$$h^1(\xi; x) := \begin{pmatrix} \xi_1 & \sqrt{-1}x_3 & \sqrt{-1}\overline{x_2} \\ \sqrt{-1}\overline{x_3} & \xi_2 & x_1 \\ \sqrt{-1}x_2 & \overline{x_1} & \xi_3 \end{pmatrix}.$$

The *exceptional Jordan algebra* $\mathcal{J}^1$ is defined by

$$\mathcal{J}^1 := (\mathcal{J}^{\mathbb{C}})_{\tau\sigma} = \{h^1(\xi; x) |\ \xi \in \mathbb{R}^3, x \in \mathbf{O}^3\}$$

with the Jordan product $X \circ Y = 2^{-1}(XY + YX)$. Then the Jordan algebra $\mathcal{J}^1$ has the trace $\mathrm{tr}(X) \in \mathbb{R}$, the identity element E of Jordan product, the inner product $(X|Y) \in \mathbb{R}$, the cross product $X \times Y \in \mathcal{J}^1$, the trilinear from $(X|Y|Z) \in \mathbb{R}$, the determinant $\det(X) \in \mathbb{R}$ and the characteristic polynomial $\Phi_X(\lambda)$ is a polynomial with $\mathbb{R}$-coefficients. Let $i \in \{1, 2, 3\}$ and $x \in \mathbf{O}$. Denote the elements

$$F_1^1(x) := F_1(x), \quad F_j^1(x) := F_j(\sqrt{-1}x) \quad \text{with } j \in \{2, 3\}.$$

Then we see

$$h^1(\xi_1, \xi_2, \xi_3; x_1, x_2, x_3) = \sum\nolimits_{i=1}^{3}(\xi_i E_i + F_i^1(x_i))$$

and for $X = \sum_{i=1}^{3}(\xi_i E_i + F_i^1(x_i)) \in \mathcal{J}^1$, denote

$$(X)_{E_i} := \xi_i = (X|E_i), \quad (X)_{F_i^1} := x_i$$

respectively. Moreover, denote the elements

$$P^+ := h^1(1, -1, 0; 0, 0, 1), \qquad P^- := h^1(-1, 1, 0; 0, 0, 1),$$
$$Q^+(x) := h^1(0, 0, 0; x, \overline{x}, 0), \qquad Q^-(x) := h^1(0, 0, 0; x, -\overline{x}, 0)$$

and the subspaces $F_i^1(\mathbf{O}) := \{F_i^1(x) |\ x \in \mathbf{O}\}$, $F_i^1(\mathrm{Im}\mathbf{O}) := \{F_i^1(p) |\ p \in \mathrm{Im}\mathbf{O}\}$, $Q^+(\mathbf{O}) := \{Q^+(x) |\ x \in \mathbf{O}\}$, $Q^-(\mathbf{O}) := \{Q^-(x) |\ x \in \mathbf{O}\}$, respectively. Easily we have the following decompositions of $\mathcal{J}^1$.

Lemma 1.5.

$$\mathcal{J}^1 = \mathbb{R}E_1 \oplus \mathbb{R}E_2 \oplus \mathbb{R}E_3 \oplus F_1^1(\mathbf{O}) \oplus F_2^1(\mathbf{O}) \oplus F_3^1(\mathbf{O}), \tag{1.5.a}$$

$$\begin{aligned}\mathcal{J}^1 = {} & \mathbb{R}(-E_1 + E_2) \oplus \mathbb{R}P^- \oplus \mathbb{R}E \oplus \mathbb{R}E_3 \oplus F_3^1(\mathrm{Im}\mathbf{O}) \\ & \oplus Q^+(\mathbf{O}) \oplus Q^-(\mathbf{O}).\end{aligned} \tag{1.5.b}$$

We use the notation $\epsilon_i(j)$ in this article. If $i = j$ then $\epsilon_i(j) := 1$ else $\epsilon_i(j) := -1$. Then $\epsilon_i(j) = (-1)^{1+\delta_{i,j}}$ where $\delta_{i,j}$ is the Kronecker delta and we write the notation $\epsilon(i)$ instead of $\epsilon_1(i)$ for short. Using Lemma 1.2, the following lemma follows from direct calculations.

Lemma 1.6. *Let $x, y \in \mathbf{O}$, $i, j \in \{1, 2, 3\}$ and indexes $i+1, i+2 \in \{1, 2, 3\}$ be counted modulo 3. Let $X = \sum_{i=1}^{3}(\xi_i E_i + F_i^1(x_i))$, $Y = \sum_{i=1}^{3}(\eta_i E_i + F_i^1(y_i)) \in \mathcal{J}^1$. Then the following equations hold:*

(1.6.a)

$$\begin{cases} \text{(i)} \;\; E_i \times E_i = 0, & \text{(ii)} \;\; E_i \times E_{i+1} = 2^{-1}E_{i+2}, \\ \text{(iii)} \;\; E_i \times F_i^1(x) = F_i^1\left(-2^{-1}x\right), & \text{(iv)} \;\; E_i \times F_j^1(x) = 0, \; i \neq j, \\ \text{(v)} \;\; F_i^1(x) \times F_i^1(y) = -\epsilon(i)(x|y)E_i, & \\ \text{(vi)} \;\; F_{i+1}^1(x) \times F_{i+2}^1(y) = F_i^1\left(-\epsilon(i)2^{-1}\overline{xy}\right), & \end{cases}$$

$$(X|Y) = \sum\nolimits_{i=1}^{3}(\xi_i\eta_i + \epsilon(i)2(x_i|y_i)), \tag{1.6.b}$$

$$\det(X) = \xi_1\xi_2\xi_3 + 2\mathrm{Re}(x_1x_2x_3) - \sum\nolimits_{i=1}^{3}\epsilon(i)\xi_i(x_i|x_i), \tag{1.6.c}$$

(1.6.d)

$$X^{\times 2} = \sum\nolimits_{i=1}^{3}((\xi_{i+1}\xi_{i+2} - \epsilon(i)(x_i|x_i))E_i + F_i^1(-\epsilon(i)\overline{x_{i+1}x_{i+2}} - \xi_i x_i)).$$

For all $X \in \mathcal{J}^1$, the *minimal subspace* V_X of X is defined by

$$V_X := \{aX^{\times 2} + bX + cE \mid a, b, c \in \mathbb{R}\}.$$

Lemma 1.7. *For all $X \in \mathcal{J}^1$, the minimal space V_X is closed under the cross product.*

Proof. It follows from (1.3.b). $\square$

The linear Lie group $\mathrm{F}_{4(-20)}$ is defined by

$$\mathrm{F}_{4(-20)} := \{g \in \mathrm{GL}_{\mathbb{R}}(\mathcal{J}^1) |\; g(X \circ Y) = gX \circ gY\}.$$

Because of $\mathcal{J}^1 = (\mathcal{J}^{\mathbb{C}})_{\tau\sigma}$, we can write $\mathrm{F}_{4(-20)} = (\mathrm{F}_4^{\mathbb{C}})^{\widetilde{\tau\sigma}}$. In [16, Theorem 2.2.2], I. Yokota shows that $\mathrm{F}_{4(-20)}$ is an exceptional linear Lie group of type $\mathbf{F}_{4(-20)}$. From Proposition 1.4, we have the following proposition.

Proposition 1.8. *Let $X, Y, Z \in \mathcal{J}^1$.*

(1) *For all $g \in \mathrm{F}_{4(-20)}$,*

$$\mathrm{tr}(gX) = \mathrm{tr}(X). \tag{1.8.a}$$

(2) *The following equations hold.*

$$
\begin{aligned}
\text{(1.8.b)}\quad \mathrm{F}_{4(-20)} &= \{g \in \mathrm{GL}_{\mathbb{R}}(\mathcal{J}^1) |\ \det(gX) = X,\ gE = E\} \\
&= \{g \in \mathrm{GL}_{\mathbb{R}}(\mathcal{J}^1) \mid \Phi_{gX}(\lambda) = \Phi_X(\lambda)\} \\
&= \{g \in \mathrm{GL}_{\mathbb{R}}(\mathcal{J}^1) |\ \det(gX) = X,\ (gX|gY) = (X|Y)\} \\
&= \left\{ g \in \mathrm{GL}_{\mathbb{R}}(\mathcal{J}^1) \ \middle|\ \begin{array}{ccc} (gX|gY|gZ) & = & (X|Y|Z), \\ (gX|gY) & = & (X|Y) \end{array} \right\} \\
&= \{g \in \mathrm{GL}_{\mathbb{R}}(\mathcal{J}^1) |\ g(X \times Y) = gX \times gY\}.
\end{aligned}
$$

By Proposition 1.8, the identity element E, the trace, the inner product, the determinant, the trilinear form, the cross product, the characteristic polynomial and the set of all characteristic roots with multiplicities are invariant under the action of $\mathrm{F}_{4(-20)}$ and we use this fact without notice.

Let $X \in \mathcal{J}^1$ and $\lambda_0 \in \mathbb{R}$. The elements $p(X),\ E_{X,\lambda_0}, W_{X,\lambda_0} \in V_X$ (see Lemma 1.7) are defined as

$$
\begin{aligned}
p(X) &:= X - 3^{-1}\mathrm{tr}(X)E, \\
E_{X,\lambda_0} &:= \mathrm{tr}((\lambda_0 E - X)^{\times 2})^{-1}(\lambda_0 E - X)^{\times 2}, \\
W_{X,\lambda_0} &:= X - \left(\lambda_0 E_{X,\lambda_0} + 2^{-1}(\mathrm{tr}(X) - \lambda_0)(E - E_{X,\lambda_0})\right)
\end{aligned}
$$

respectively. Immediately, we have the following lemma.

Lemma 1.9. *Let $X \in \mathcal{J}^1$. If $\mathrm{tr}((\lambda_0 E - X)^{\times 2}) \neq 0$ then E_{X,λ_0} and W_{X,λ_0} are well-defined and the following equation holds.*

$$
\text{(1.9)}\qquad X = \lambda_0 E_{X,\lambda_0} + 2^{-1}(\mathrm{tr}(X) - \lambda_0)(E - E_{X,\lambda_0}) + W_{X,\lambda_0}.
$$

Lemma 1.10. *Let $X \in \mathcal{J}^1$. For all $g \in \mathrm{F}_{4(-20)}$,*

$$
\text{(1.10)}\qquad \left\{ \begin{array}{llll} \text{(i)} & g(V_X) = V_{gX}, & \text{(ii)} & gp(X) = p(gX), \\ \text{(iii)} & gE_{X,\lambda_1} = E_{gX,\lambda_1}, & \text{(iv)} & gW_{X,\lambda_1} = W_{gX,\lambda_1}. \end{array} \right.
$$

Proof. It follows from Proposition 1.8. □

2. The principle of triality.

In this section, we explain the groups Spin(8) and Spin(7) by means of the triality. In the next section, we explain that these groups are isomorphic to some stabilizers in $\mathrm{F}_{4(-20)}$, respectively.

We write S^0 for the subset $\{\mathrm{diag}(1, 1, \cdots, 1), \mathrm{diag}(-1, 1, \cdots, 1)\} \cong \mathbb{Z}_2$ of $M(n, \mathbb{R})$ and SO(0) for the group $\{1\}$, respectively.

Lemma 2.1. (cf. [17, 18]). *Let n be a natural number, and p, q non-negative integers with $p + q > 0$.*

(1) (cf. [17, Theorem 20(2)]). $\mathrm{O}(n) = S^0 \ltimes \mathrm{SO}(n)$. *Especially,* $\mathrm{O}^0(n) = \mathrm{SO}(n)$.

(2) (cf. [18, Theorem 6.12(2)]). $\mathrm{O}^0(p, q) \simeq (\mathrm{SO}(p) \times \mathrm{SO}(q)) \times \mathbb{R}^{pq}$.

(3) *For all* $n \geq 3$, *the fundamental group* $\pi_1(\mathrm{O}^0(n,1)) = \mathbb{Z}_2$

Proof. (3) We note $\pi_1(\mathrm{SO}(n)) = \mathbb{Z}_2$ $(n \geq 3)$ (cf. [17, Theorem 59(2)]). By (2), $\pi_1(\mathrm{O}^0(n,1)) = \pi_1(\mathrm{SO}(n)) \times \pi_1(\mathrm{SO}(0)) \times \pi_1(\mathbb{R}^{pq}) = \mathbb{Z}_2$. $\square$

From now on, the groups $\mathrm{O}(8)$, $\mathrm{SO}(8)$, $\mathrm{O}(7)$ and $\mathrm{SO}(7)$ are identified with the groups: $\mathrm{O}(8) = \{g \in \mathrm{GL}_{\mathbb{R}}(\mathbf{O}) \mid (gx|gy) = (x|y)\}$, $\mathrm{SO}(8) = \{g \in \mathrm{GL}_{\mathbb{R}}(\mathbf{O}) \mid (gx|gy) = (x|y),\ \det(g) = 1\}$, $\mathrm{O}(7) = \{g \in \mathrm{O}(8) \mid g1 = 1\}$, $\mathrm{SO}(7) = \{g \in \mathrm{SO}(8) \mid g1 = 1\}$, respectively. The element $\epsilon \in \mathrm{O}(8)$ is defined by

$$\epsilon x := \overline{x} \quad \text{for } x \in \mathbf{O}.$$

Then $\epsilon^2 = 1$ and its determinant is $-1 : \det(\epsilon) = -1$. The involutive automorphism t of the group $\mathrm{SO}(8)^3$ is defined by

$$t(g_1, g_2, g_3) := (g_1, g_2, \epsilon g_3 \epsilon) \quad \text{for } (g_1, g_2, g_3) \in \mathrm{SO}(8)^3.$$

The subgroup $T(\mathbf{O})$ of $\mathrm{SO}(8)^3$ is defined by

$$T(\mathbf{O}) := \{(g_1, g_2, g_3) \in \mathrm{SO}(8)^3 \mid (g_1x)(g_2y) = g_3(xy) \text{ for all } x, y \in \mathbf{O}\}$$

(cf. [4, (2.4.6)], [11], [13], [19]) and the subgroup $\tilde{D}_4$ of $\mathrm{SO}(8)^3$ by

$$\begin{aligned}\tilde{D}_4 &:= t^{-1}(T(\mathbf{O})) = \{(g_1, g_2, g_3) \in \mathrm{SO}(8)^3 \mid t(g_1, g_2, g_3) \in T(\mathbf{O})\} \\ &= \{(g_1, g_2, g_3) \in \mathrm{SO}(8)^3 \mid (g_1x)(g_2y) = \epsilon g_3 \epsilon(xy) \text{ for all } x, y \in \mathbf{O}\}.\end{aligned}$$

The equation $(g_1x)(g_2y) = g_3(xy)$ or $(g_1x)(g_2y) = \epsilon g_3\epsilon(xy)$ is called the *triality*. The following result is proved in [11] (cf. [19, Lemma 1.14.3]).

Lemma 2.2. *Let* $i \in \{1, 2, 3\}$ *and indexes* $i, i+1, i+2$ *be counted modulo* 3. *Assume that* $(g_1, g_2, g_3) \in \mathrm{O}(8)^3$ *satisfies* $(g_ix)(g_{i+1}y) = \epsilon g_{i+2}\epsilon(xy)$ *for all* $x, y \in \mathbf{O}$. *Then* $(g_{i+1}x)(g_{i+2}y) = \epsilon g_i \epsilon(xy)$ *for all* $x, y \in \mathbf{O}$. *Especially,*

$$(2.2) \qquad (g_1, g_2, g_3) \in \tilde{D}_4 \Leftrightarrow (g_2, g_3, g_1) \in \tilde{D}_4 \Leftrightarrow (g_3, g_1, g_2) \in \tilde{D}_4.$$

For $i \in \{1, 2, 3\}$, the homomorphism $p_i : \tilde{D}_4 \to \mathrm{SO}(8)$ is defined by

$$p_i(g_1, g_2, g_3) := g_i \qquad \text{for } (g_1, g_2, g_3) \in \tilde{D}_4.$$

Lemma 2.3. *Let* $x, y \in \mathbf{O}$.

(1) *Let* $g \in \mathrm{SO}(7)$. *Then*

$$(2.3.\text{a}) \qquad \text{(i) } \overline{gx} = g\overline{x}, \quad \text{(ii) } \epsilon g \epsilon = g, \quad \text{(iii) } g(\mathrm{Im}(x)) = \mathrm{Im}(gx).$$

Especially, $g(\mathrm{Im}\mathbf{O}) \subset \mathrm{Im}\mathbf{O}$.

(2) *Let* $(g_1, g_2, g_3) \in \tilde{D}_4$, $i \in \{1, 2, 3\}$ *and indexes* $i, i+1, i+2$ *be counted modulo* 3. *Then*

$$(2.3.\text{b}) \qquad g_i 1 = 1 \ \Leftrightarrow\ g_{i+1} = \epsilon g_{i+2} \epsilon \ \Leftrightarrow\ g_{i+2} = \epsilon g_{i+1} \epsilon.$$

(3) *Assume that* $(g_1, g_2, g_3) \in \tilde{D}_4$ *and* $g_3 1 = 1$. *Then*

(2.3.c)

$$\begin{cases} \text{(i)} \ \ g_3(x\overline{y}) = (g_1x)(\overline{g_1y}), \quad \text{(ii)} \ \ g_3(\mathrm{Im}(x\overline{y})) = \mathrm{Im}((g_1x)(\overline{g_1y})), \\ \text{(iii)} \ \ g_1(xy) = (g_3x)(g_1y). \end{cases}$$

Proof. (1) Because of $g1 = 1$, $\overline{x} = 2(1|x) - x$ and $\mathrm{Im}(x) = x - (1|x)$, we see $g\overline{x} = 2(1|gx) - gx = \overline{gx}$. and $g(\mathrm{Im}(x)) = g(x - (1|x)) = gx - (1|gx) = \mathrm{Im}(gx)$. Thus (i) and (iii) follow. From (i), $\epsilon g \epsilon x = g(\overline{\overline{x}}) = gx$.

(2) Obviously $g_{i+1} = \epsilon g_{i+2} \epsilon$ iff $g_{i+2} = \epsilon g_{i+1} \epsilon$. We show $g_i 1 = 1$ iff $g_{i+1} = \epsilon g_{i+2} \epsilon$. By (2.2), $(g_1, g_2, g_3) \in \tilde{D}_4$ iff $(g_i, g_{i+1}, g_{i+2}) \in \tilde{D}_4$ so that

(i) $$(g_i x)(g_{i+1} y) = \epsilon g_{i+2} \epsilon (xy) \quad \text{for all } x, y \in \mathbf{O}.$$

Suppose $g_i 1 = 1$. Substituting $x = 1$ in (i), we obtain $g_{i+1} = \epsilon g_{i+2} \epsilon$. Conversely, suppose $g_{i+1} = \epsilon g_{i+2} \epsilon$. Substituting $x = y = 1$ in (i), $(g_i 1)(\epsilon g_{i+2} \epsilon 1) = \epsilon g_{i+2} \epsilon 1$. Multiplying $(\epsilon g_{i+2} \epsilon 1)^{-1}$ from right, $g_i 1 = 1$.

(3) By (2.3.b), $g_2 = \epsilon g_1 \epsilon$ and $g_1 = \epsilon g_2 \epsilon$. First, because of $g_3 = \epsilon g_3 \epsilon$ (by (2.3.a)(ii)) and $(g_1, g_2, g_3) = (g_1, \epsilon g_1 \epsilon, g_3) \in \tilde{D}_4$, we see that $g_3(x\overline{y}) = \epsilon g_3 \epsilon (x\overline{y}) = (g_1 x)(\epsilon g_1 \epsilon \overline{y}) = (g_1 x)(\overline{g_1 y})$. Second, by (2.3.a)(iii) and (2.3.c)(i), we see that $g_3(\mathrm{Im}(x\overline{y})) = \mathrm{Im}(g_3(x\overline{y})) = \mathrm{Im}((g_1 x)(\overline{g_1 y}))$. Last, because of $g_1 = \epsilon g_2 \epsilon$ and $(g_3, g_1, g_2) \in \tilde{D}_4$ (by (2.2)), we obtain $g_1(xy) = \epsilon g_2 \epsilon (xy) = (g_3 x)(g_1 y)$. $\square$

The subgroup $\tilde{B}_3$ of $\tilde{D}_4$ is defined by

$$\begin{aligned} \tilde{B}_3 &:= \{(g_1, g_2, g_3) \in \tilde{D}_4 |\ g_3 1 = 1\} \\ &= \{(g_1, g_2, g_3) \in \tilde{D}_4 |\ g_2 = \epsilon g_1 \epsilon\} = \{(g_1, g_2, g_3) \in \tilde{D}_4 \mid g_1 = \epsilon g_2 \epsilon\} \end{aligned}$$

and the homomorphism $q : \tilde{B}_3 \to \mathrm{SO}(7)$ by $q := p_3 | \tilde{B}_3$: $q(g_1, g_2, g_3) = g_3$. The linear Lie group G_2 is defined by

$$\mathrm{G}_2 := \mathrm{Aut}(\mathbf{O}) = \{g \in \mathrm{GL}_{\mathbb{R}}(\mathbf{O}) \mid (gx)(gy) = g(xy)\}.$$

Then G_2 is a subgroup of $\mathrm{SO}(7)$ (cf. [19, Lemma 1.2.1, Theorem 1.9.3]). In particular, $(gx|gy) = (x|y)$ and $g1 = 1$ for all $g \in \mathrm{G}_2$. Also, for any $g \in \mathrm{G}_2$, considering $(g, g, g) \in \mathrm{SO}(8)^3$, G_2 is a subgroup of $\tilde{B}_3$. Now $S^7 = \{a \in \mathbf{O} |\ \mathrm{n}(a) = 1\}$ and $S^6 = \{a \in \mathrm{Im}\mathbf{O} |\ \mathrm{n}(a) = 1\}$. For all $a \in S^7$, the elements $L_a, R_a, T_a \in \mathrm{GL}_{\mathbb{R}}(\mathbf{O})$ are defined by

$$L_a x := ax,\ R_a x := xa,\ T_a x := axa \quad \text{for } x \in \mathbf{O}$$

respectively. By (1.1.b), $L_a, R_a, T_a \in \mathrm{O}(8)$. Because S^7 is connected and for the unite $1 \in \mathbf{O}$, $L_1 = R_1 = T_1 = 1_{\mathbf{O}}$ where $1_{\mathbf{O}}$ denotes the identity element of $\mathrm{O}(8)$, we obtain $L_a, R_a, T_a \in \mathrm{O}^0(8) = \mathrm{SO}(8)$ (see Lemma 2.1(1)). For any $a_i \in S^7$, denote the elements $L_{a_n, \cdots, a_1}$, $R_{a_n, \cdots, a_1}$, $T_{a_n, \cdots, a_1} \in \mathrm{SO}(8)$ as

$$\begin{cases} L_{a_n, \cdots, a_1} := L_{a_n} \cdots L_{a_1}, \quad R_{a_n, \cdots, a_1} := R_{a_n} \cdots R_{a_1}, \\ \qquad\qquad T_{a_n, \cdots, a_1} := T_{a_n} \cdots T_{a_1}. \end{cases}$$

Lemma 2.4. (1) (cf. [19, Theorem 1.9.1,Theorem 1.9.2]).

(2.4.a) $$S^6 = Orb_{\mathrm{G}_2}(e_1),$$

$$(\mathrm{G}_2)_{e_1} \cong \mathrm{SU}(3), \quad \mathrm{G}_2/\mathrm{SU}(3) \simeq S^6.$$

Furthermore, G_2 *is connected.*

(2) *If* $a_i \in S^7$, *then* $(L_{a_n,\cdots,a_1}, R_{a_n,\cdots,a_1}, \epsilon T_{a_n,\cdots,a_1}\epsilon) \in \tilde{D}_4$.

(3) *If* $a, b \in S^6$, *then* $(L_{b,a}, R_{b,a}, T_{b,a}) \in \tilde{B}_3$.

(4) *Let* $j \in \{1, 2\}$.

(2.4.b)
$$\mathrm{G}_2 = \{(g_1, g_2, g_3) \in \tilde{B}_3 |\ g_j 1 = 1\}.$$

Proof. (2) Let $a \in S^7$. By (1.1.k),

$$(L_a x)(R_a y) = T_a(xy) = \epsilon(\epsilon T_a \epsilon)\epsilon(xy).$$

Because of $L_a,\ R_a,\ \epsilon T_a \epsilon \in \mathrm{SO}(8)$, we see $(L_a, R_a, \epsilon T_a \epsilon) \in \tilde{D}_4$. Next, from the induction, it follows that

$$(L_{a_n,\cdots,a_1} x)(R_{a_n,\cdots,a_1} y) = \epsilon(\epsilon T_{a_n,\cdots,a_1}\epsilon)\epsilon(xy).$$

Hence (2) follows.

(3) From (2), $(L_{b,a}, R_{b,a}, \epsilon T_{b,a}\epsilon) \in \tilde{D}_4$. Because of $a^2 = b^2 = -1$, we see $\epsilon T_{b,a}\epsilon 1 = \overline{b(a(\overline{1})a)b} = 1$. Then $\epsilon T_{b,a}\epsilon \in \mathrm{SO}(7)$ and by (2.3.a)(ii), $\epsilon T_{b,a}\epsilon = T_{b,a}$. Hence (3) follows.

(4) Suppose that $(g_1, g_2, g_3) \in \tilde{B}_3$. Because of $g_2 = \epsilon g_1 \epsilon$, we see that $g_1 1 = 1$ iff $g_2 1 = 1$. Thus it is enough to show $\mathrm{G}_2 = \{(g_1, g_2, g_3) \in \tilde{B}_3 |\ g_1 1 = 1\}$. Put $G = \{(g_1, g_2, g_3) \in \tilde{B}_3 |\ g_1 1 = 1\}$. Immediately, $\mathrm{G}_2 \subset G$. Conversely, take $g = (g_1, g_2, g_3) \in G$. Because of $g \in G$, we see $g_1 1 = g_2 1 = g_3 1 = 1$. Because of $g_i 1 = g_{i+1} 1 = 1$ and (2.3.b), we see $g_{i+1} = \epsilon g_{i+2} \epsilon$ and $g_{i+2} = \epsilon g_i \epsilon$. Then $g_{i+1} = \epsilon g_{i+2} \epsilon = \epsilon(\epsilon g_i \epsilon)\epsilon = g_i$. Moving $i \in \{1, 2, 3\}$, $g_1 = g_2 = g_3$. Thus $(g_1, g_2, g_3) \in \mathrm{G}_2$ and so $G \subset \mathrm{G}_2$. Hence $G = \mathrm{G}_2$. □

Lemma 2.5. (1) $\tilde{B}_3/\mathrm{G}_2 \simeq S^7$. *Furthermore,* $\tilde{B}_3$ *is connected.*

(2) $\tilde{D}_4/\tilde{B}_3 \simeq S^7$. *Furthermore,* $\tilde{D}_4$ *is connected.*

Proof. (1) We consider the action of $\tilde{B}_3$ on S^7 as $x \mapsto p_1(g_1, g_2, g_3)x = g_1 x$ for $x \in S^7$ and $(g_1, g_2, g_3) \in \tilde{B}_3$. Fix $x \in S^7$. Then x can be expressed by

$$x = \cos\theta + a \sin\theta \quad \text{for some } a \in S^6 \text{ and } \theta \in \mathbb{R}.$$

First, by (2.4.a), there exists $g_1 \in \mathrm{G}_2$ such that $g_1 a = e_1$. Obviously $g_1 1 = 1$ and we set $h_1 = (g_1, g_1, g_1) \in \mathrm{G}_2 \subset \mathrm{Spin}(7)$. Then

$$p_1(h_1)x = \cos\theta + e_1 \sin\theta.$$

Second, put $h_{e_1,e_2} = (L_{e_1,e_2}, R_{e_1,e_2}, T_{e_1,e_2})$. Because of $e_i \in S^6$ and Lemma 2.4(3), we see $h_{e_1,e_2} \in \tilde{B}_3$ and

$$p_1(h_{e_1,e_2})p_1(h_1)x = e_1(e_2(\cos\theta + e_1 \sin\theta)) = e_3 \cos\theta + e_2 \sin\theta.$$

Third, because of $e_3 \cos\theta + e_2 \sin\theta \in S^6$, there exists $g_2 \in \mathrm{G}_2$ such that $g_2(e_3 \cos\theta + e_2 \sin\theta) = e_1$. Then letting $h_2 = (g_2, g_2, g_2) \in \mathrm{G}_2 \subset \mathrm{Spin}(7)$,

$$p_1(h_2)p_1(h_{e_1,e_2})p_1(h_1)x = e_1.$$

Last, letting $h_{e_3,e_2} = (L_{e_3,e_2}, R_{e_3,e_2}, T_{e_3,e_2}) \in \tilde{B}_3$,

$$p_1(h_{e_3,e_2})p_1(h_2)p_1(h_{e_1,e_2})p_1(h_1)x = e_3(e_2e_1) = 1.$$

Hence $\tilde{B}_3$ acts transitively on S^7. By (2.4.b), $(\tilde{B}_3)_1 = \mathrm{G}_2$. Thus $\tilde{B}_3/\mathrm{G}_2 \simeq S^7$ follows. Since G_2 and S^7 are connected, $\tilde{B}_3$ is also connected.

(2) We consider the action of $\tilde{D}_4$ on S^7 as $x \mapsto p_3(g_1, g_2, g_3)x = g_3x$ for $x \in S^7$ and $(g_1, g_2, g_3) \in \tilde{D}_4$. Let $x \in S^7$. Because of $(\overline{x}|\overline{x}) = 1$, $\overline{x} \in S^7$. By Lemma 2.4(2), $(L_{\overline{x}}, R_{\overline{x}}, \epsilon T_{\overline{x}}\epsilon) \in \tilde{D}_4$. Then it follows from (2.2) that $(R_{\overline{x}}, \epsilon T_{\overline{x}}\epsilon, L_{\overline{x}}) \in \tilde{D}_4$ and

$$p_3(R_{\overline{x}}, \epsilon T_{\overline{x}}\epsilon, L_{\overline{x}})x = \overline{x}x = 1.$$

Thus $\tilde{D}_4$ acts transitively on S^7. Because of $(\tilde{D})_1 = \tilde{B}_3$, we see $\tilde{D}_4/\tilde{B}_3 \simeq S^7$. Since $\tilde{B}_3$ and S^7 are connected, $\tilde{D}_4$ is also connected. $\square$

The following result is shown in [4] (cf. [19, Theorems 1.16.2, 1.15.2]).

Proposition 2.6. (1) *The following sequence is exact:*

$$1 \to \{(1,1,1), (\epsilon_i(1), \epsilon_i(2), \epsilon_i(3))\} \to \tilde{D}_4 \xrightarrow{p_i} \mathrm{SO}(8) \to 1. \tag{2.6.a}$$

(2) *The following sequence is exact:*

$$1 \to \{(1,1,1), (-1,-1,1)\} \to \tilde{B}_3 \xrightarrow{q} \mathrm{SO}(7) \to 1. \tag{2.6.b}$$

By Lemma 2.5(2) and (2.6.a), $\tilde{D}_4$ is connected and a two-hold covering group of SO(8), and by Lemma 2.5(1) and (2.6.b), $\tilde{B}_3$ is connected and a two-hold covering group of SO(7). So denote

$$\mathrm{Spin}(8) := \tilde{D}_4, \quad \mathrm{Spin}(7) := \tilde{B}_3.$$

3. The construction of concrete elements of $\mathrm{F}_{4(-20)}$.

In order to give the orbit decomposition of $\mathcal{J}^1$ under the action of $\mathrm{F}_{4(-20)}$, we must know concrete elements of $\mathrm{F}_{4(-20)}$ and these operation on $\mathcal{J}^1$. In this section, we present concrete elements $\varphi_0(g_1, g_2, g_3)$ and $\exp(t\tilde{A}_i^1(a))$.

Lemma 3.1. *The following equations hold.*

$$(\mathrm{F}_{4(-20)})_{F_3^1(1)} = (\mathrm{F}_{4(-20)})_{E_3,F_3^1(1)}. \tag{3.1.a}$$

$$(\mathrm{F}_{4(-20)})_{E_1,F_3^1(1)} = (\mathrm{F}_{4(-20)})_{E_1,E_2,E_3,F_3^1(1)} = (\mathrm{F}_{4(-20)})_{E_2,F_3^1(1)}. \tag{3.1.b}$$

Proof. For all $g \in (\mathrm{F}_{4(-20)})_{F_3^1(1)}$, $gE_3 = g(F_3^1(1)^{\times 2}) = (gF_3^1(1))^{\times 2} = F_3^1(1)^{\times 2} = E_3$. Thus $g \in (\mathrm{F}_{4(-20)})_{F_3^1(1),E_3}$ and so $(\mathrm{F}_{4(-20)})_{F_3^1(1)} \subset (\mathrm{F}_{4(-20)})_{F_3^1(1),E_3}$. From $(\mathrm{F}_{4(-20)})_{F_3^1(1),E_3} \subset (\mathrm{F}_{4(-20)})_{F_3^1(1)}$, (3.1.a) follows. Next, obviously, $(\mathrm{F}_{4(-20)})_{E_1,E_2,E_3,F_3^1(1)} \subset (\mathrm{F}_{4(-20)})_{E_1,F_3^1(1)}$. Conversely, fix $g \in (\mathrm{F}_{4(-20)})_{E_1,F_3^1(1)}$. By (3.1.a), $g \in (\mathrm{F}_{4(-20)})_{E_1,E_3,F_3^1(1)}$ and then $gE_2 = g(E - (E_1 + E_3)) = E_2$. Thus $g \in (\mathrm{F}_{4(-20)})_{E_1,E_2,E_3,F_3^1(1)}$ and so

$(\mathrm{F}_{4(-20)})_{E_1,F_3^1(1)} \subset (\mathrm{F}_{4(-20)})_{E_1,E_2,E_3,F_3^1(1)}$. Hence $(\mathrm{F}_{4(-20)})_{E_1,E_2,E_3,F_3^1(1)} = (\mathrm{F}_{4(-20)})_{E_1,F_3^1(1)}$. Similarly, $(\mathrm{F}_{4(-20)})_{E_1,E_2,E_3,F_3^1(1)} = (\mathrm{F}_{4(-20)})_{E_2,F_3^1(1)}$. $\square$

Lemma 3.2. (1) *The group* $\mathrm{Spin}(8)$ *is isomorphic to the stabilizer* $(\mathrm{F}_{4(-20)})_{E_1,E_2,E_3}$ *of* $\mathrm{F}_{4(-20)}$, *that is, the isomorphism* $\varphi_0 : \mathrm{Spin}(8) \to (\mathrm{F}_{4(-20)})_{E_1,E_2,E_3}$ *is given by*

$$\varphi_0(g_1, g_2, g_3)\Big(\sum\nolimits_{i=1}^{3}(\xi_i E_i + F_i^1(x_i))\Big) = \sum\nolimits_{i=1}^{3}(\xi_i E_i + F_i^1(g_i x_i)). \tag{3.2}$$

(2) *The restriction of* φ_0 *on the subgroup* $\mathrm{Spin}(7)$ *is an isomorphism from* $\mathrm{Spin}(7)$ *onto* $(\mathrm{F}_{4(-20)})_{E_1,E_2,E_3,F_3^1(1)}$.

Proof. (1) We can prove it similar to [19, Theorem 2.7.1].

(2) By (1), φ_0 is a mono-morphism. Thus it is enough to show that φ_0 is onto. Fix $g \in \mathrm{B}_3$. By (1), $g = \varphi_0(g_1, g_2, g_3)$ for some $(g_1, g_2, g_3) \in \mathrm{Spin}(8)$. Then $F_3^1(1) = \varphi_0(g_1, g_2, g_3)F_3^1(1) = F_3^1(g_3 1)$. Thus $g_3 1 = 1$ and so $(g_1, g_2, g_3) \in \mathrm{Spin}(7)$. Hence $\varphi_0 : \mathrm{Spin}(7) \to (\mathrm{F}_{4(-20)})_{E_1,E_2,E_3,F_3^1(1)}$ is onto. $\square$

Denote the subgroups $\mathrm{D}_4 := \varphi_0(\mathrm{Spin}(8)) = (\mathrm{F}_{4(-20)})_{E_1,E_2,E_3}$ and $\mathrm{B}_3 := \varphi_0(\mathrm{Spin}(7)) = (\mathrm{F}_{4(-20)})_{E_1,E_2,E_3,F_3^1(1)}$, respectively.

Lemma 3.3. *Let* $g = (g_1, g_2, g_3) \in \mathrm{Spin}(7)$ *and* $p, x \in \mathbf{O}$.

$$\begin{cases} \text{(i)} \ \ \varphi_0(g)(-E_1 + E_2) = -E_1 + E_2, \\ \text{(ii)} \ \ \varphi_0(g)P^- = P^-, \quad \text{(iii)} \ \ \varphi_0(g)F_3^1(p) = F_3^1(g_3 p), \\ \text{(iv)} \ \ \varphi_0(g)E_3 = E_3, \qquad \text{(v)} \ \ \varphi_0(g)E = E, \\ \text{(vi)} \ \ \varphi_0(g)Q^+(x) = Q^+(g_1 x), \\ \text{(vii)} \ \ \varphi_0(g)Q^-(x) = Q^-(g_1 x). \end{cases} \tag{3.3}$$

Proof. From $\varphi_0(g) \in \mathrm{B}_3 = (\mathrm{F}_{4(-20)})_{E_1,E_2,E_3,F_3^1(1)}$, the first five equations follow. Because of $g_2 = \epsilon g_1 \epsilon$, we see $\varphi_0(g)F_2^1(\overline{x}) = F_2^1(\epsilon g_1 \epsilon \overline{x}) = F_2^1(\overline{g_1 x})$. Thus (vi) and (vii) follow. $\square$

Proposition 3.4. *Let* $Y = \mathrm{diag}(r_1, r_2, r_3) \in \mathcal{J}^1$ *where* r_1, r_2, r_3 *are different from each other. Then* $(\mathrm{F}_{4(-20)})_Y = \mathrm{D}_4$.

Proof. Obviously $\mathrm{D}_4 = (\mathrm{F}_{4(-20)})_{E_1,E_2,E_3} \subset (\mathrm{F}_{4(-20)})_Y$. Conversely, fix $g \in (\mathrm{F}_{4(-20)})_Y$. Let $i \in \{1, 2, 3\}$ and indexes $i, i+1, i+2$ are counted modulo 3. Because of $(r_i E - Y)^{\times 2} = (r_{i+1} - r_i)(r_{i+2} - r_i)E_i$ and $\mathrm{tr}((r_i E - Y)^{\times 2}) = (r_{i+1} - r_i)(r_{i+2} - r_i) \neq 0$, we see that E_{Y,r_i} is well-defined and $E_{Y,r_i} = E_i$. By (1.10)(iii), $gE_i = gE_{Y,r_i} = E_{gY,r_i} = E_{Y,r_i} = E_i$. Thus $g \in \mathrm{D}_4$ and so $(\mathrm{F}_{4(-20)})_Y \subset \mathrm{D}_4$. Hence $(\mathrm{F}_{4(-20)})_Y = \mathrm{D}_4$. $\square$

Denote the Lie algebra $\mathfrak{f}_{4(-20)} := Lie(\mathrm{F}_{4(-20)})$ (*resp.* $\mathfrak{f}_4^{\mathbb{C}} := Lie(\mathrm{F}_4^{\mathbb{C}})$). By Proposition 1.8 (*resp.* Proposition 1.4),

$$\mathfrak{f}_{4(-20)} = \{\delta \in \mathrm{End}_{\mathbb{R}}(\mathcal{J}^1) \mid \delta(X \times Y) = \delta X \times Y + X \times \delta Y\}$$
$$(\textit{resp. } \mathfrak{f}_4^{\mathbb{C}} = \{\delta \in \mathrm{End}_{\mathbb{C}}(\mathcal{J}^{\mathbb{C}}) \mid \delta(X \times Y) = \delta X \times Y + X \times \delta Y\}).$$

Given $\delta \in \mathfrak{f}_{4(-20)}$ (*resp.* $\mathfrak{f}_4^{\mathbb{C}}$),

$$\operatorname{tr}(\delta X) = 0,\ \delta E = 0,\ (\delta X|Y) + (X|\delta Y) = 0,\ (\delta X|X|X) = 0$$

for all $X, Y \in \mathcal{J}^1$ (*resp.* $\mathcal{J}^{\mathbb{C}}$). The Lie subalgebra $\mathfrak{d}_4$ (*resp.* $\mathfrak{d}_4^{\mathbb{C}}$) of $\mathfrak{f}_{4(-20)}$ (*resp.* $\mathfrak{f}_4^{\mathbb{C}}$) is defined by

$$\mathfrak{d}_4 := \{D \in \mathfrak{f}_{4(-20)}|\ DE_i = 0,\ i = 1, 2, 3\}$$
$$(\textit{resp.}\ \mathfrak{d}_4^{\mathbb{C}} := \{D \in \mathfrak{f}_4^{\mathbb{C}}|\ DE_i = 0,\ i = 1, 2, 3\})$$

and the Lie algebra $\mathfrak{D}_4$ by

$$\mathfrak{D}_4 := \{D \in \operatorname{End}_{\mathbb{R}}(\mathbf{O})|\ (Dx|y) + (x|Dy) = 0\}.$$

The following lemma is proved in [4](cf. [19, Lemmas 1.3.6, 1.3.7, Proposition 2.3.7]).

Lemma 3.5. *Let $x, y \in \mathbf{O}$.*

(1) *For all $D_1 \in \mathfrak{D}_4$, there exist $D_2, D_3 \in \mathfrak{D}_4$ such that*

$$(D_1x)y + x(D_2y) = \epsilon D_3\epsilon(xy).$$

Also such D_2 and D_3 are uniquely determined for D_1.

(2) *For $D_1,\ D_2,\ D_3 \in \mathfrak{D}_4$, the relation*

$$(D_1x)y + x(D_2y) = \epsilon D_3\epsilon(xy)$$

implies that

$$(D_2x)y + x(D_3y) = \epsilon D_1\epsilon(xy), \quad (D_3x)y + x(D_1y) = \epsilon D_2\epsilon(xy).$$

(3) *The Lie algebra $\mathfrak{d}_4$ is isomorphic to the Lie algebra $\mathfrak{D}_4$ under the correspondence $D_1 \mapsto d\varphi_0(D_1, D_2, D_3)$ given by*

$$d\varphi_0(D_1, D_2, D_3)\Big(\sum_{i=1}^{3}(\xi_iE_i + F_i^1(x_i))\Big) = \sum_{i=1}^{3} F_i^1(D_ix_i). \tag{3.5}$$

where D_2 and D_3 are elements of $\mathfrak{D}_4$ which are determined by D_1 from the infinitesimal triality:

$$(D_1x)y + x(D_2y) = \epsilon D_3\epsilon(xy).$$

Let $i \in \{1, 2, 3\}$ and indexes $i, i+1, i+2$ be counted modulo 3. For $a \in \mathbf{O}^{\mathbb{C}}$, the skew-hermitian matrix $A_i(a) \in M(3, \mathbf{O}^{\mathbb{C}})$ is denoted by

$$A_i(a) := aE_{i+1,i+2} - \overline{a}E_{i+2,i+1},$$

the linear subspace $\mathfrak{u}_i^{\mathbb{C}}$ of $M(3, \mathbf{O}^{\mathbb{C}})$ by $\mathfrak{u}_i^{\mathbb{C}} := \{A_i(a)|\ a \in \mathbf{O}^{\mathbb{C}}\}$ and the linear subspace $\mathfrak{R}^{\mathbb{C}}$ of $M(3, \mathbf{O}^{\mathbb{C}})$ by

$$\mathfrak{R}^{\mathbb{C}} := \mathfrak{u}_1^{\mathbb{C}} \oplus \mathfrak{u}_2^{\mathbb{C}} \oplus \mathfrak{u}_3^{\mathbb{C}} = \{A \in M(3, \mathbf{O}^{\mathbb{C}})|\ A^* = -A,\ \operatorname{diag}(A) = 0\}$$

where $\operatorname{diag}(A) = 0$ means that all diagonal elements a_{ii} of A are 0. For $A \in \mathfrak{R}^{\mathbb{C}}$, the element $\tilde{A} \in \operatorname{End}_{\mathbb{C}}(\mathcal{J}^{\mathbb{C}})$ is defined by

$$\tilde{A}X := AX - XA \quad \text{for } X \in \mathcal{J}^{\mathbb{C}}$$

the linear subspaces $\tilde{\mathfrak{u}}_i^{\mathbb{C}}$ of $\mathrm{End}_{\mathbb{C}}(\mathcal{J}^{\mathbb{C}})$ by $\tilde{\mathfrak{u}}_i^{\mathbb{C}} := \{\tilde{A}_i(a) |\ a \in \mathbf{O}^{\mathbb{C}}\}$ and the linear subspace $\tilde{\mathfrak{R}}^{\mathbb{C}}$ of $\mathrm{End}_{\mathbb{C}}(\mathcal{J}^{\mathbb{C}})$ by

$$\tilde{\mathfrak{R}}^{\mathbb{C}} := \{\tilde{A} |\ A \in \mathfrak{R}^{\mathbb{C}}\} = \tilde{\mathfrak{u}}_1^{\mathbb{C}} \oplus \tilde{\mathfrak{u}}_2^{\mathbb{C}} \oplus \tilde{\mathfrak{u}}_3^{\mathbb{C}}.$$

By direct calculations, we have the following lemma.

Lemma 3.6. *Let $a \in \mathbf{O}^{\mathbb{C}}$, $i \in \{1, 2, 3\}$ and indexes $i, i+1, i+2$ be counted modulo 3. Then the operation of $\tilde{A}_i(a)$ on $\mathcal{J}^{\mathbb{C}}$ is given by*

$$(3.6) \quad \begin{cases} \text{(i)} \quad \tilde{A}_i(a)E_i = 0, \\ \text{(ii)} \quad \tilde{A}_i(a)E_{i+1} = F_i(-a), \quad \text{(iii)} \quad \tilde{A}_i(a)E_{i+2} = F_i(a), \\ \text{(iv)} \quad \tilde{A}_i(a)F_i(x) = 2(a|x)(E_{i+1} - E_{i+2}), \\ \text{(v)} \quad \tilde{A}_i(a)F_{i+1}(x) = F_{i+2}(\overline{ax}), \\ \text{(vi)} \quad \tilde{A}_i(a)F_{i+2}(x) = F_{i+1}(-\overline{xa}). \end{cases}$$

The following lemma is proved in [4](cf. [19, Proposition 2.3.6, Theorem 2.3.8]).

Lemma 3.7. (1) *$\tilde{\mathfrak{R}}^{\mathbb{C}}$ is a $\mathbb{C}$-linear subspace of $\mathfrak{f}_4^{\mathbb{C}}$.*

(2) *Any element $\delta \in \mathfrak{f}_4^{\mathbb{C}}$ is uniquely expressed by*

$$\delta = D + \tilde{A} \quad \textit{for some } D \in \mathfrak{d}_4^{\mathbb{C}} \textit{ and } \tilde{A} \in \tilde{\mathfrak{R}}^{\mathbb{C}}.$$

Especially,

$$(3.7) \qquad \mathfrak{f}_4^{\mathbb{C}} = \mathfrak{d}_4^{\mathbb{C}} \oplus \tilde{\mathfrak{R}}^{\mathbb{C}} = \mathfrak{d}_4^{\mathbb{C}} \oplus \tilde{\mathfrak{u}}_1^{\mathbb{C}} \oplus \tilde{\mathfrak{u}}_2^{\mathbb{C}} \oplus \tilde{\mathfrak{u}}_3^{\mathbb{C}}.$$

We denote the differential of the involutive automorphism $\widetilde{\tau\sigma}$ of $\mathrm{F}_4^{\mathbb{C}}$ as same notation $\widetilde{\tau\sigma}$. Looking again $\mathfrak{f}_4^{\mathbb{C}}$ as an $\mathbb{R}$-Lie algebra, the involutive $\mathbb{R}$-automorphism $\widetilde{\tau\sigma}$ of $\mathfrak{f}_4^{\mathbb{C}}$ induces the $\mathbb{R}$-Lie subalgebra $(\mathfrak{f}_4^{\mathbb{C}})_{\widetilde{\tau\sigma}}$ of $\mathfrak{f}_4^{\mathbb{C}}$. Then we can write $\mathfrak{f}_{4(-20)} = (\mathfrak{f}_4^{\mathbb{C}})_{\widetilde{\tau\sigma}} = \{\delta \in \mathfrak{f}_4^{\mathbb{C}} |\ \tau\sigma\delta\sigma\tau = \delta\}$.

Lemma 3.8. *The following equations hold.*

$$(3.8) \quad \begin{cases} \text{(i)} \quad (\mathfrak{d}_4^{\mathbb{C}})_{\widetilde{\tau\sigma}} = \mathfrak{d}_4, \\ \text{(ii)} \quad (\tilde{\mathfrak{u}}_1^{\mathbb{C}})_{\widetilde{\tau\sigma}} = \{\tilde{A}_1(a) |\ a \in \mathbf{O}\}, \\ \text{(iii)} \quad (\tilde{\mathfrak{u}}_j^{\mathbb{C}})_{\widetilde{\tau\sigma}} = \{\tilde{A}_j(\sqrt{-1}a) |\ a \in \mathbf{O}\} \quad \textit{with } j = 2, 3. \end{cases}$$

Proof. Easily, (i) follows. Suppose that $\tilde{A}_i(z) \in (\tilde{\mathfrak{u}}_i^{\mathbb{C}})_{\widetilde{\tau\sigma}}$ with $z \in \mathbf{O}^{\mathbb{C}}$. From (3.6), we see $\tau\sigma\tilde{A}_i(z)\sigma\tau E_{i+1} = \epsilon(i)F_i(-\tau z)$ and $\tilde{A}_i(z)E_{i+1} = \epsilon(i)F_i(-z)$, so that $\epsilon(i)\tau z = z$. Thus, if $i = 1$ then $z \in \mathbf{O}$, else $z = \sqrt{-1}a$ for some $a \in \mathbf{O}$. Therefore, we have the necessary conditions of (ii) and (iii). Conversely, put $S = \{E_i, F_i^1(x) |\ x \in \mathbf{O},\ i = 1, 2, 3\}$. S spans the linear space of $\mathcal{J}^{\mathbb{C}}$ over $\mathbb{C}$. Let $X \in S$ and $a \in \mathbf{O}$. Because of (3.6) and $\tau\sigma X = X$, we see that $\tau\sigma\tilde{A}_1(a)\sigma\tau X = \tilde{A}_1(a)X$ and $\tau\sigma\tilde{A}_j(\sqrt{-1}a)\sigma\tau X = \tilde{A}_j(\sqrt{-1}a)X$ $(j = 2, 3)$. Hence the sufficient condition follows. $\square$

For $a \in \mathbf{O}$, skew-hermitian matrix $A_i^1(a) \in M(3, \mathbf{O}^{\mathbb{C}})$ is defined by

$$A_1^1(a) := A_1(a), \quad A_j(a) := A_j(\sqrt{-1}a) \quad \text{with } j \in \{2,3\}.$$

The element $\tilde{A}_i^1(a) \in \mathfrak{f}_{4(-20)}$ is defined by

$$\tilde{A}_i^1(a)X := A_i^1(a)X - XA_i^1(a) \quad \text{for } X \in \mathcal{J}^1$$

and the linear subspace $\tilde{\mathfrak{u}}_i^1$ of $\mathfrak{f}_{4(-20)}$ by $\tilde{\mathfrak{u}}_i^1 := \{\tilde{A}_i^1(a) |\ a \in \mathbf{O}\}$. Then we have the following lemma.

Lemma 3.9. (1) *The following equation holds.*

$$\mathfrak{f}_{4(-20)} = \mathfrak{d}_4 \oplus \tilde{\mathfrak{u}}_1^1 \oplus \tilde{\mathfrak{u}}_2^1 \oplus \tilde{\mathfrak{u}}_3^1. \tag{3.9.a}$$

(2) *Let $a \in \mathbf{O}$, $i \in \{1,2,3\}$ and indexes $i, i+1, i+2$ be counted modulo 3. Then the operation of $\tilde{A}_i^1(a)$ on $\mathcal{J}^1$ is given by*

$$\begin{cases} \text{(i)} \quad \tilde{A}_i^1(a)E_i = 0, \\ \text{(ii)} \quad \tilde{A}_i^1(a)E_{i+1} = F_i^1(-a), \quad \text{(iii)} \quad \tilde{A}_i^1(a)E_{i+2} = F_i^1(a), \\ \text{(iv)} \quad \tilde{A}_i^1(a)F_i^1(x) = \epsilon(i)2(a|x)(E_{i+1} - E_{i+2}), \\ \text{(v)} \quad \tilde{A}_i^1(a)F_{i+1}^1(x) = F_{i+2}^1(-\epsilon(i+2)\overline{ax}), \\ \text{(vi)} \quad \tilde{A}_i^1(a)F_{i+2}^1(x) = F_{i+1}^1(\epsilon(i+1)\overline{xa}). \end{cases} \tag{3.9.b}$$

Lemma 3.10. *Let $t \in \mathbb{R}$, $a \in S^7$, $\xi, \eta \in \mathbb{R}^3$ and $x, y \in \mathbf{O}^3$. Let $i \in \{1,2,3\}$ and indexes $i, i+1, i+2$ be counted modulo 3.*

(1) *When $i = 1$, let $h^1(\eta; y) \in \mathcal{J}^1$ be*

$$\begin{cases} \eta_1 &= \xi_1, \\ \eta_2 &= 2^{-1}((\xi_2 + \xi_3) + (\xi_2 - \xi_3)\cos 2t) + (a|x_1)\sin 2t, \\ \eta_3 &= 2^{-1}((\xi_2 + \xi_3) - (\xi_2 - \xi_3)\cos 2t) - (a|x_1)\sin 2t, \\ y_1 &= x_1 - 2^{-1}(\xi_2 - \xi_3)a\sin 2t - 2(a|x_1)a\sin^2 t, \\ y_2 &= x_2\cos t - \overline{x_3 a}\sin t, \\ y_3 &= x_3\cos t + \overline{a x_2}\sin t \end{cases} \tag{3.10.a}$$

and when $i \in \{2,3\}$, let $h^1(\eta; y) \in \mathcal{J}^1$ be

(3.10.b)

$$\begin{cases} \eta_i &= \xi_i, \\ \eta_{i+1} &= 2^{-1}((\xi_{i+1} + \xi_{i+2}) + (\xi_{i+1} - \xi_{i+2})\cosh 2t) - (a|x_i)\sinh 2t, \\ \eta_{i+2} &= 2^{-1}((\xi_{i+1} + \xi_{i+2}) - (\xi_{i+1} - \xi_{i+2})\cosh 2t) + (a|x_i)\sinh 2t, \\ y_i &= x_i - 2^{-1}(\xi_{i+1} - \xi_{i+2})a\sinh 2t + 2(a|x_i)a\sinh^2 t. \\ y_{i+1} &= x_{i+1}\cosh t + \overline{x_{i+2} a}\sinh t, \\ y_{i+2} &= x_{i+2}\cosh t + \overline{a x_i}\sinh t. \end{cases}$$

Then $h^1(\eta; y) = \exp(t\tilde{A}_i^1(a))h^1(\xi; x)$ and $\exp(t\tilde{A}_i^1(a)) \in (\mathrm{F}_{4(-20)})^0_{E_i}$.

(2) *If $a \in S^6$, then $\exp(t\tilde{A}_i^1(a)) \in (\mathrm{F}_{4(-20)})^0_{F_i^1(1)}$ for all $i \in \{1,2,3\}$.*

Proof. (1) Fix $i \in \{1,2,3\}$. Let $F : \mathbb{R} \times \mathcal{J}^1 \to \mathcal{J}^1$ be the mapping defined by $F(t, h^1(\xi; x)) = h^1(\eta; y)$. From direct calculations, we have

$$\frac{d}{dt}F(t, h^1(\xi; x)) = \tilde{A}^1_i(a)F(t, h^1(\xi; x)) \text{ and } F(0, h^1(\xi; x)) = h^1(\xi; x)$$

and it follows from the uniqueness of solutions that $F(t, h^1(\xi; x)) = \exp(t\tilde{A}^1_i(a))h^1(\xi; x)$. Because of $\eta_i = \xi_i$, we see $\exp(t\tilde{A}^1_j(a))E_i = E_i$ and therefore $\exp(t\tilde{A}^1_i(a)) \in (\mathrm{F}_{4(-20)})^0_{E_i}$. Hence (1) follow.

(2) Because of (1) and $(a|1) = 0$, we see $\exp(t\tilde{A}^1_i(a))F^1_i(1) = F^1_i(1)$. Hence (3) follows. □

We give elementarily two lemmas which implies the difference of orbits of the elements in $\mathcal{J}^1$ under the action of $\mathrm{F}_{4(-20)}$.

Fix $Y \in \mathcal{J}^1$. The inner product B_Y on $\mathcal{J}^1$ is defined by

$$B_Y(X_1, X_2) = (Y|X_1|X_2) \quad \text{for } X_i \in \mathcal{J}^1.$$

Lemma 3.11. *Let $Y_1, Y_2 \in \mathcal{J}^1$. Assume that B_{Y_1} and B_{Y_2} have different signatures from each other. Then $Orb_{\mathrm{F}_{4(-20)}}(Y_1) \neq Orb_{\mathrm{F}_{4(-20)}}(Y_2)$. Furthermore,*

$$Orb_{\mathrm{F}_{4(-20)}}(E_1) \neq Orb_{\mathrm{F}_{4(-20)}}(E_2) = Orb_{\mathrm{F}_{4(-20)}}(E_3), \tag{3.11.a}$$

$$Orb_{\mathrm{F}_{4(-20)}}(E_1 - E_2) \neq Orb_{\mathrm{F}_{4(-20)}}(-E_1 + E_2). \tag{3.11.b}$$

Proof. Suppose that there exists $g \in \mathrm{F}_{4(-20)}$ such that $gY_1 = Y_2$. Then

$$B_{Y_1}(X_1, X_2) = (Y_1|X_1|X_2) = (gY_1|gX_1|gX_2) = B_{Y_2}(gX_1, gX_2)$$

for all $X_1, X_2 \in \mathcal{J}^1$. Using Sylvester's theorem, inner products B_{Y_1} and B_{Y_2} have the same signature. It contradicts with the assumption. Thus $Orb_{\mathrm{F}_{4(-20)}}(Y_1) \neq Orb_{\mathrm{F}_{4(-20)}}(Y_2)$. Let $Y \in \{E_1,\ E_2,\ E_1 - E_2,\ -E_1 + E_2\}$. For any $X = \sum_{i=1}^3(\xi_i E_i + F^1_i(x_i)) \in \mathcal{J}^1$, we have the following table:

Y	$B_Y(X, X)$	The signature of B_Y
E_1	$\xi_2\xi_3 - (x_1\|x_1)$	$(9, 1)$
E_3	$\xi_1\xi_2 + (x_3\|x_3)$	$(1, 9)$
$E_1 - E_2$	$\xi_2\xi_3 - \xi_3\xi_1 - (x_1\|x_1) - (x_2\|x_2)$	$(18, 2)$
$-E_1 + E_2$	$-\xi_2\xi_3 + \xi_3\xi_1 + (x_1\|x_1) + (x_2\|x_2)$	$(2, 18)$

Then we see $Orb_{\mathrm{F}_{4(-20)}}(E_1) \neq Orb_{\mathrm{F}_{4(-20)}}(E_3)$ and $Orb_{\mathrm{F}_{4(-20)}}(E_1 - E_2) \neq Orb_{\mathrm{F}_{4(-20)}}(-E_1 + E_2)$. Now from (3.10.a), $\exp(2^{-1}\pi\tilde{A}^1_1(1))E_3 = E_2$ and $Orb_{\mathrm{F}_{4(-20)}}(E_2) = Orb_{\mathrm{F}_{4(-20)}}(E_3)$. Hence the result follows. □

Denote the linear subspace $(\mathcal{J}^1)_0$ of $\mathcal{J}^1$ as

$$(\mathcal{J}^1)_0 := \{X \in \mathcal{J}^1 |\ \mathrm{tr}(X) = 0\},$$

and the subsets $\mathcal{R}^{\pm}$ of $\mathcal{J}^1$ as

$$\mathcal{R}^+ := \{X \in (\mathcal{J}^1)_0 |\ X^{\times 2} = P^+\}, \quad \mathcal{R}^- := \{X \in (\mathcal{J}^1)_0 |\ X^{\times 2} = P^-\}$$

respectively.

Lemma 3.12. (1) *For all* $X \in \mathcal{R}^{\pm}$, $\mathrm{tr}(X) = \mathrm{tr}(X^{\times 2}) = \det(X) = 0$. *Furthermore,* $(X^{\times 2}) \times X = 0$.

(2) *The following equations hold:*

$$\mathcal{R}^{+} = \emptyset, \tag{3.12.a}$$

$$\mathcal{R}^{-} = \{rP^{-} + Q^{+}(x) |\ r \in \mathbb{R},\ x \in S^7\}. \tag{3.12.b}$$

Furthermore,

$$Orb_{\mathrm{F}_{4(-20)}}(P^{+}) \neq Orb_{\mathrm{F}_{4(-20)}}(P^{-}). \tag{3.12.c}$$

Proof. (1) Because of $X \in \mathcal{R}^{\pm}$, we see $\mathrm{tr}(X) = 0 = \mathrm{tr}(P^{\pm}) = \mathrm{tr}(X^{\times 2})$, and from (1.3.b)(iv), we see $\det(X)X = (X^{\times 2})^{\times 2} = (P^{\pm})^{\times 2} = 0$. Thus $\mathrm{tr}(X) = \mathrm{tr}(X^{\times 2}) = \det(X) = 0$ and by (1.3.b)(v), $(X^{\times 2}) \times X = 0$.

(2) Suppose that there exists $X = \sum_{i=1}^{3}(\xi_i E_i + F_i^1(x_i)) \in \mathcal{R}^{+}$. Using (1.6.a), from (1), we see

$$\begin{aligned} 0 = (X^{\times 2}) \times X = P^{+} \times X = 2^{-1}\big(&-\xi_3 E_1 + \xi_3 E_2 \\ &+ (\xi_2 - \xi_1 + 2(1|x_3))E_3 + F_1^1(x_1 - \overline{x_2}) + F_2^1(-x_2 - \overline{x_1}) + F_3^1(-\xi_3)\big). \end{aligned}$$

Then $\xi_3 = 0$. However, by (1.6.d),

$$1 = (P^{+})_{E_1} = (X^{\times 2})_{E_1} = \xi_2 \cdot 0 - (x_1|x_1) = -(x_1|x_1) \leq 0.$$

It is a contradiction and (3.12.a) follows.

Next, put $\mathcal{R} = \{rP^{-} + Q^{+}(x) |\ r \in \mathbb{R},\ x \in S^7\}$. Take $X = \sum_{i=1}^{3}(\xi_i E_i + F_i^1(x_i)) \in \mathcal{R}^{-}$. From (1),

$$\begin{aligned} 0 = (X^{\times 2}) \times X = P^{-} \times X = 2^{-1}\big(&\xi_3 E_1 - \xi_3 E_2 \\ &+ (-\xi_2 + \xi_1 + 2(1|x_3))E_3 + F_1^1(x_1 - \overline{x_2}) + F_2^1(x_2 - \overline{x_1}) + F_3^1(-\xi_3)\big). \end{aligned}$$

Then $\xi_3 = 0$ and $x_2 = \overline{x_1}$. Because of $\xi_1 + \xi_2 = \mathrm{tr}(X) = 0$, we see $\xi_2 = -\xi_1$. Next, by (1.6.d),

$$\begin{aligned} P^{-} = X^{\times 2} = &-(x_1|x_1)E_1 + (x_1|x_1)E_2 + (-\xi_1^2 + (x_3|x_3))E_3 \\ &+ F_1^1(-\overline{\overline{x_1}x_3} - \xi_1 x_1) + F_2^1(\overline{x_3 x_1} + \xi_1 \overline{x_1}) + F_3^1(\mathrm{n}(x_1)). \end{aligned}$$

Then $\mathrm{n}(x_1) = 1$ and $0 = x_1(\overline{x_3} + \xi_1)$. From $x_1 \neq 0$, we see $x_3 = -\xi_1$ and $X = -\xi_1 P^{-} + F_1^1(x_1) + F_2^1(\overline{x_1})$ where $\mathrm{n}(x_1) = 1$. Thus $X \in \mathcal{R}$ and so $\mathcal{R}^{-} \subset \mathcal{R}$. Conversely, let $X = rP^{-} + Q^{+}(x) \in \mathcal{R}$ where $r \in \mathbb{R}$ and $x \in S^7$. From direct calculations, we see $X \in \mathcal{R}^{-}$ and so $\mathcal{R} \subset \mathcal{R}^{-}$. Hence $\mathcal{R}^{-} = \mathcal{R}$.

Last, we show (3.12.c). Suppose that there exists $g \in \mathrm{F}_{4(-20)}$ such that $gP^{+} = P^{-}$. From (3.12.a) and (3.12.b), we see $\emptyset = g(\mathcal{R}^{+}) = \mathcal{R}^{-} \neq \emptyset$. It is a contradiction as required. $\square$

4. The stabilizers of Spin group type.

In this section, we will explain the construction of the spin groups $\mathrm{Spin}(9)$, $\mathrm{Spin}^0(8,1)$ and $\mathrm{Spin}^0(7,1)$ as the stabilizers, respectively.

For $X \in \mathcal{J}^1$, denote the element $L^\times(X) \in \mathrm{End}_{\mathbb{R}}(\mathcal{J}^1)$ by

$$L^\times(X)Y := X \times Y \quad \text{for } Y \in \mathcal{J}^1$$

and for $r \in \mathbb{R}$, consider the r-eigenspace of $L^\times(X)$ on $\mathcal{J}^1$:

$$\mathcal{J}^1_{L^\times(X),r} = \{Y \in \mathcal{J}^1 |\ L^\times(X)Y = rY\}.$$

The quadratic form Q on $\mathcal{J}^1$ is defined by

$$Q(X) := -\mathrm{tr}(X^{\times 2}) \quad \text{for } X \in \mathcal{J}^1.$$

Lemma 4.1. *Let $i \in \{1,2,3\}$ and indexes $i, i+1, i+2$ be counted modulo 3. Let $X \in \mathcal{J}^1$ and $r \in \mathbb{R}$. Then*

$$Q(gX) = Q(X) \quad \textit{for all } g \in \mathrm{F}_{4(-20)}, \tag{4.1.a}$$

$$g\mathcal{J}^1_{L^\times(X),r} = \mathcal{J}^1_{L^\times(gX),r} \quad \textit{for all } g \in \mathrm{F}_{4(-20)}, \tag{4.1.b}$$

$$\mathcal{J}^1_{L^\times(2E_i),-1} = \{\xi(E_{i+1} - E_{i+2}) + F^1_i(x) |\ \xi \in \mathbb{R},\ x \in \mathbf{O}\}, \tag{4.1.c}$$

$$Q(\xi(E_{i+1} - E_{i+2}) + F^1_i(x)) = \xi^2 + \epsilon(i)(x|x). \tag{4.1.d}$$

Especially, when $i = 1$, then the quadratic space $(\mathcal{J}^1_{L^\times(2E_1),-1}, Q)$ is isomorphic to $(\mathbb{R}^{0,9}, \mathrm{q}_{0,9})$, and when $i \in \{2,3\}$, then $(\mathcal{J}^1_{L^\times(2E_i),-1}, Q)$ is isomorphic to $(\mathbb{R}^{8,1}, \mathrm{q}_{8,1})$.

Proof. Using Proposition 1.8 and Lemma 1.6, it follows from direct calculations. □

The quadratic space $(\mathcal{J}^1_{L^\times(2E_1),-1}, Q)$ has a sphere S^8 as

$$S^8 := \{X \in \mathcal{J}^1_{L^\times(2E_1),-1} \mid Q(X) = 1\}$$

and the quadratic space $(\mathcal{J}^1_{L^\times(2E_3),-1}, Q)$ has a positive sphere $\mathcal{S}^{8,1}$, a negative sphere $\mathcal{S}^{8,1}(-1)$ and a null cone $\mathcal{N}^{8,1}$ as

$$\begin{aligned}\mathcal{S}^{8,1} &:= \{X \in \mathcal{J}^1_{L^\times(2E_3),-1} \mid Q(X) = 1\},\\ \mathcal{S}^{8,1}(-1) &:= \{X \in \mathcal{J}^1_{L^\times(2E_3),-1} \mid Q(X) = -1\},\\ \mathcal{N}^{8,1} &:= \{X \in \mathcal{J}^1_{L^\times(2E_3),-1} \mid Q(X) = 0,\ X \neq 0\}\end{aligned}$$

respectively. Denote the subsets $\mathcal{S}^{8,1}_+,\ \mathcal{S}^{8,1}_- \subset \mathcal{S}^{8,1}$ by

$$\mathcal{S}^{8,1}_+ := \{X \in \mathcal{S}^{8,1} \mid (X|E_1) > 0\},\ \mathcal{S}^{8,1}_- := \{X \in \mathcal{S}^{8,1} \mid (X|E_1) < 0\}$$

respectively, and the subsets $\mathcal{N}^{8,1}_+,\ \mathcal{N}^{8,1}_- \subset \mathcal{N}^{8,1}$ by

$$\mathcal{N}^{8,1}_+ := \{X \in \mathcal{N}^{8,1} \mid (X|E_1) > 0\},\ \mathcal{N}^{8,1}_- := \{X \in \mathcal{N}^{8,1} \mid (X|E_1) < 0\}$$

respectively.

Lemma 4.2. *The following equations holds.*

$$\mathcal{S}^{8,1} = \mathcal{S}^{8,1}_{+} \coprod \mathcal{S}^{8,1}_{-}, \tag{4.2.a}$$

$$\mathcal{N}^{8,1} = \mathcal{N}^{8,1}_{+} \coprod \mathcal{N}^{8,1}_{-}. \tag{4.2.b}$$

Proof. Suppose that $X \in \mathcal{J}^1_{L^\times(2E_3),-1}$ satisfies $X \neq 0$ and $(E_1|X) = 0$. From (4.1.c), we see $X = F^1_3(x)$ for some $(0 \neq)x \in \mathbf{O}$ and $Q(X) < 0$. Therefore, if $X \in \mathcal{S}^{8,1}$ (resp. $X \in \mathcal{N}^{8,1}$), then $(E_1|X) \neq 0$. $\square$

The quadratic subspace $(\mathcal{J}^1_{7,1}, Q)$ is defined by

$$\begin{aligned}\mathcal{J}^1_{7,1} &:= \{X \in \mathcal{J}^1_{L^\times(2E_3),-1} |\ (F^1_3(1)|X) = 0\} \\ &= \{\xi(E_1 - E_2) + F^1_3(x) \mid \xi \in \mathbb{R},\ x \in \mathrm{Im}\mathbf{O}\}\end{aligned}$$

and $Q(\xi(E_1-E_2)+F^1_3(x)) = \xi^2-(x|x)$. So the quadratic space $(\mathcal{J}^1_{7,1}, Q)$ is isomorphic to $(\mathbb{R}^{7,1}, \mathrm{q}_{7,1})$ and we denote a positive sphere $\mathcal{S}^{7,1}$ in the quadratic space $(\mathcal{J}^1_{7,1}, Q)$ and its subset $\mathcal{S}^{7,1}_{+}$ as

$$\mathcal{S}^{7,1} := \{X \in \mathcal{J}^1_{7,1} \mid Q(X) = 1\},\ \mathcal{S}^{7,1}_{+} := \{X \in \mathcal{S}^{7,1} \mid (X|E_1) > 0\}$$

respectively. Denote the homomorphisms $\tilde{p}_i, \tilde{q}, p_i, q$ as

$$\begin{aligned}
\tilde{p}_i &: (\mathrm{F}_{4(-20)})_{E_i} \to \mathrm{O}(\mathcal{J}^1_{L^\times(2E_i),-1}, Q), && \tilde{p}_i(g) := g|\mathcal{J}^1_{L^\times(2E_i),-1}, \\
\tilde{q} &: (\mathrm{F}_{4(-20)})_{F^1_3(1)} \to \mathrm{O}(\mathcal{J}^1_{7,1}, Q), && \tilde{q}(g) := g|\mathcal{J}^1_{7,1}, \\
p_i &: \mathrm{D}_4 \to \mathrm{O}(F^1_i(\mathbf{O}), Q), && p_i(g) := g|F^1_i(\mathbf{O}), \\
q &: \mathrm{B}_3 \to \mathrm{O}(F^1_3(\mathrm{Im}\mathbf{O}), Q), && q(g) := g|F^1_3(\mathrm{Im}\mathbf{O})
\end{aligned}$$

respectively.

Lemma 4.3. *The homomorphisms* $\tilde{p}_i,\ \tilde{q},\ p,\ q$ *are well-defined.*

Proof. First, by (4.1.b), $(\mathrm{F}_{4(-20)})_{E_i}$ invariants $\mathcal{J}^1_{L^\times(2E_i),-1}$. Second, because of (3.1.a), the definition of $\mathcal{J}^1_{7,1}$ and (4.1.b), $(\mathrm{F}_{4(-20)})_{F^1_3(1)}$ invariants $\mathcal{J}^1_{7,1}$. Third, because of $F^1_i(\mathbf{O}) = \{X \in \mathcal{J}^1 |\ E_j \times X = 0,\ i \neq j\}$, $\mathrm{D}_4 = (\mathrm{F}_{4(-20)})_{E_1,E_2,E_3}$ invariants $F^1_i(\mathbf{O})$. Last, because of $F^1_i(\mathrm{Im}\mathbf{O}) = \{X \in F^1_i(\mathbf{O}) \mid (F^1_i(1)|X) = 0\}$, $\mathrm{B}_3 = (\mathrm{F}_{4(-20)})_{E_1,E_2,E_3,F^1_3(1)}$ invariants $F^1_3(\mathrm{Im}\mathbf{O})$. Therefore, from (4.1.a), it follows that the restrictions of suitable subspaces of $\mathcal{J}^1$ induce the homomorphisms into suitable orthogonal groups. $\square$

We use trivial lemma to determine the G-orbits of X.

Lemma 4.4. *Let X be a set and a group G act on X. Let I be an index set and $i, j \in I$. Assume that there exists a sequence $(X_i)_{i\in I}$ of subsets of X and a sequence $(v_i)_{i\in I}$ of elements in X such that the following conditions* (i)-(iv) *hold:*
(i) $X = \cup_{i\in I} X_i$, (ii) $v_i \in X_i$, (iii) $Orb_G(v_i) \neq Orb_G(v_j) \Leftrightarrow i \neq j$,
(iv) $X_i \subset Orb_G(v_i)$.

Then $X_i = Orb_G(v_i)$ *for all* $i \in I$.

Proof. Take $x \in Orb_G(v_i)$. Because of $x \in Orb_G(v_i) \subset X = \cup_{i\in I} X_i$, there exists $j \in I$ such that $x \in X_j$. By (iv), $x \in X_j \subset Orb_G(v_j)$ so that $x \in Orb_G(v_i) \cap Orb_G(v_j)$. By (iii), $i = j$ and $x \in X_i$. Then $Orb_G(v_i) \subset X_i$. Thus, from (iv), we have $X_i = Orb_G(v_i)$. $\square$

Lemma 4.5. *Let* $j \in \{1, 2, 3\}$. *For all* $X = \sum_{i=1}^{3}(\xi_i E_i + F_i^1(x_i)) \in \mathcal{J}^1$, *there exists* $\varphi_0(g_1, g_2, g_3) \in \mathrm{D}_4$ *such that*
(4.5)

$$\varphi_0(g_1, g_2, g_3)X = \left(\sum{}_{i=1}^{3}\xi_i E_i\right) + F_j^1\left(\sqrt{\mathrm{n}(x_j)}\right) + \sum{}_{k=1}^{2} F_{j+k}^1(g_{j+k}x_{j+k})$$

where the index $j + k$ *is counted modulo* 3.

Proof. Given $x_j = (X)_{F_j^1} \in \mathbf{O}$, we can take $g_j \in SO(8)$ such that $g_j x_j = \sqrt{\mathrm{n}(x_j)}$. By (2.6.a), there exists $(g_1, g_2, g_3) \in \tilde{D}_4$ and from (3.2), we have the result. $\square$

Lemma 4.6. *The following equations hold.*

(4.6.a) $\qquad S^8 = Orb_{(\mathrm{F}_{4(-20)})_{E_1}}(E_2 - E_3).$

(4.6.b) $\qquad \mathcal{S}^{8,1}(-1) = Orb_{(\mathrm{F}_{4(-20)})_{E_3}}(F_3^1(1)).$

(4.6.c) $\qquad \begin{cases} \text{(i)} & \mathcal{S}^{8,1}_+ = Orb_{(\mathrm{F}_{4(-20)})_{E_3}}(E_1 - E_2), \\ \text{(ii)} & \mathcal{S}^{8,1}_- = Orb_{(\mathrm{F}_{4(-20)})_{E_3}}(-E_1 + E_2). \end{cases}$

(4.6.d) $\qquad \begin{cases} \text{(i)} & \mathcal{N}^{8,1}_+ = Orb_{(\mathrm{F}_{4(-20)})_{E_3}}(P^+), \\ \text{(ii)} & \mathcal{N}^{8,1}_- = Orb_{(\mathrm{F}_{4(-20)})_{E_3}}(P^-). \end{cases}$

Furthermore, $\mathcal{S}^{8,1}_+$ *is connected.*

Proof. (a) By Lemma 4.3, $(\mathrm{F}_{4(-20)})_{E_1}$ acts on S^8. Fix $X \in S^8$. By (4.1.c) and (4.1.d), X can be expressed by $X = \xi(E_2 - E_3) + F_1^1(x)$ where $\xi \in \mathbb{R}$, $x \in \mathbf{O}$ and $\xi^2 + \mathrm{n}(x) = 1$. By (4.5), there exists $g \in \mathrm{D}_4 \subset (\mathrm{F}_{4(-20)})_{E_1}$ such that $gX = \xi(E_2 - E_3) + F_1^1(\sqrt{\mathrm{n}(x)})$. We can write

$$gX = \cos 2t(E_2 - E_3) + F_1^1(\sin 2t)$$

for some $t \in \mathbb{R}$. For this t, using (3.10.a), $\exp(t\tilde{A}_1^1(1)) \in (\mathrm{F}_{4(-20)})_{E_1}$ and from direct calculations, we see that

$$\exp(t\tilde{A}_1^1(1))gX = E_2 - E_3.$$

Hence (4.6.a) follows.

(b) By Lemma 4.3, $(\mathrm{F}_{4(-20)})_{E_3}$ acts on $\mathcal{S}^{8,1}(-1)$. Fix $X \in \mathcal{S}^{8,1}(-1)$. By (4.1.c) and (4.1.d), X can be expressed by $X = \xi(E_1 - E_2) + F_3^1(x)$ where $\xi \in \mathbb{R}$, $x \in \mathbf{O}$ and $\xi^2 - \mathrm{n}(x) = -1$. By (4.5), there exists $g \in \mathrm{D}_4 \subset (\mathrm{F}_{4(-20)})_{E_3}$ such that

$$gX = \xi(E_1 - E_2) + F_3^1(r_0) \quad \text{where } r_0 \geq 0,\ \xi^2 - r_0^2 = -1.$$

By (3.10.b), $\exp\left(4^{-1}\log\left((r_0+\xi)/(r_0-\xi)\right)\tilde{A}_3^1(1)\right) \in (\mathrm{F}_{4(-20)})_{E_3}$ and because of $\xi^2 - r_0^2 = -1$, we calculate that

$$\exp\left(4^{-1}\log\left((r_0+\xi)/(r_0-\xi)\right)\tilde{A}_3^1(1)\right) gX = F_3^1(\pm 1).$$

When $F_3^1(-1)$, multiplying $\varphi_0(1,-1,-1) \in \mathrm{D}_4$ from the left side,

$$\varphi_0(1,-1,-1)\exp\left(4^{-1}\log\left((r_0+\xi)/(r_0-\xi)\right)\tilde{A}_3^1(1)\right) gX = F_3^1(1).$$

Hence (4.6.b) follows.

(c) We show (4.6.c) by using Lemma 4.4. Denote $G = (\mathrm{F}_{4(-20)})_{E_3}$ or $(\mathrm{F}_{4(-20)})^0_{E_3}$. By Lemma 4.3, G acts on $\mathcal{S}^{8,1}$. We consider that $X = \mathcal{S}^{8,1}$, $(X_1, X_2) = (\mathcal{S}^{8,1}_+, \mathcal{S}^{8,1}_-)$, $(v_1, v_2) = (E_1 - E_, - E_1 + E_2)$ in Lemma 4.4. First, the condition (i) follows from (4.2.a). Second, the condition (ii) follows from direct calculations. Third, by (3.11.b), the condition (iii) follows from

$$\begin{aligned} &Orb_{(\mathrm{F}_{4(-20)})_{E_3}}(E_1 - E_2) \subset Orb_{\mathrm{F}_{4(-20)}}(E_1 - E_2) \\ \neq &Orb_{\mathrm{F}_{4(-20)}}(E_1 - E_2) \supset Orb_{(\mathrm{F}_{4(-20)})_{E_3}}(E_1 - E_2). \end{aligned}$$

Last, we show the condition (iv). Take $X \in \mathcal{S}^{8,1}_+$. By (4.5), there exists $g \in \mathrm{D}_4 \subset G$ such that

$$gX = \xi(E_1 - E_2) + F_3^1(r_0) \quad \text{where } \xi > 0,\ r_0 \geq 0,\ \xi^2 - r_0^2 = 1.$$

By (3.10.b), $\exp\left(4^{-1}\log\left((\xi+r_0)/(\xi-r_0)\right)\tilde{A}_3^1(1)\right) \in G$ and because of $\xi > 0$ and $\xi^2 - r_0^2 = 1$, we calculate that

$$\exp\left(4^{-1}\log\left((\xi+r_0)/(\xi-r_0)\right)\tilde{A}_3^1(1)\right) gX = E_1 - E_2.$$

Thus $X \in Orb_G(E_1 - E_2)$ and so $\mathcal{S}^{8,1}_+ \subset Orb_G(E_1 - E_2)$. Next, take $X \in \mathcal{S}^{8,1}_-$. By (4.5), there exists $g \in \mathrm{D}_4 \subset G$ such that

$$gX = \xi(E_1 - E_2) + F_3^1(r_0) \quad \text{where } \xi < 0,\ r_0 \geq 0,\ \xi^2 - r_0^2 = 1.$$

By (3.10.b), $\exp\left(4^{-1}\log\left((\xi+r_0)/(\xi-r_0)\right)\tilde{A}_3^1(1)\right) \in G$ and because of $\xi < 0$ and $\xi^2 - r_0^2 = 1$, we calculate that

$$\exp\left(4^{-1}\log\left((\xi+r_0)/(\xi-r_0)\right)\tilde{A}_3^1(1)\right) gX = -E_1 + E_2.$$

Thus $X \in Orb_G(-E_1 + E_2)$ and so $\mathcal{S}^{8,1}_- \subset Orb_G(-E_1 + E_2)$. Therefore, the condition (iv) follows. Hence (4.6.c) follows from Lemma 4.4. Furthermore, since $\mathcal{S}^{8,1}_+$ is a orbit of one element $E_1 - E_2$ under the action of a connected group $(\mathrm{F}_{4(-20)})^0_{E_3}$, $\mathcal{S}^{8,1}_+$ is connected.

(d) We show (4.6.d) by using Lemma 4.4. Denote $G = (\mathrm{F}_{4(-20)})_{E_3}$. Since G acts on $\mathcal{N}^{8,1}$, we consider $X = \mathcal{N}^{8,1}$, $(X_1, X_2) = (\mathcal{N}^{8,1}_+, \mathcal{N}^{8,1}_-)$, $(v_1, v_2) = (P^+, P^-)$ in Lemma 4.4. First, the condition (i) follows

from (4.2.b). Second, the condition (ii) follows from direct calculations. Third, by (3.12.c), the condition (iii) follows from

$$\begin{aligned} &Orb_{(\mathrm{F}_{4(-20)})_{E_3}}(P^+) \subset Orb_{\mathrm{F}_{4(-20)}}(P^+) \\ \neq &Orb_{\mathrm{F}_{4(-20)}}(P^-) \supset Orb_{(\mathrm{F}_{4(-20)})_{E_3}}(P^-). \end{aligned}$$

Last, we will show the condition (iv). Take $X \in \mathcal{N}_+^{8,1}$. Because of (4.5), there exists $g \in \mathrm{D}_4 \subset G$ such that

$$gX = \xi(E_1 - E_2) + F_3^1(\xi) \quad \text{where } \xi > 0.$$

By (3.10.b), $\exp\left(2^{-1}(\log\xi)\tilde{A}_3^1(1)\right) \in G$ and because of $\xi > 0$, we calculate that

$$\exp\left(2^{-1}(\log\xi)\tilde{A}_3^1(1)\right) gX = P^+.$$

Thus $X \in Orb_G(P^+)$, so that $\mathcal{N}_+^{8,1} \subset Orb_G(P^+)$. Next, take $X \in \mathcal{N}_-^{8,1}$. Because of (4.5), there exists $g \in \mathrm{D}_4 \subset G$ such that

$$gX = \xi(-E_1 + E_2) + F_3^1(\xi) \quad \text{where } \xi > 0.$$

By (3.10.b), $\exp\left(2^{-1}(\log\xi)\tilde{A}_3^1(1)\right) \in G$ and because of $\xi > 0$, we calculate that

$$\exp\left(2^{-1}(\log\xi)\tilde{A}_3^1(1)\right) gX = P^-.$$

Thus $X \in Orb_G(P^-)$ and so $\mathcal{N}_-^{8,1} \subset Orb_G(P^-)$. Therefore, the condition (iv) follows. Hence (4.6.d) follows from Lemma 4.4. $\square$

Lemma 4.7. (1) $(\mathrm{F}_{4(-20)})_{E_1}/\mathrm{D}_4 \simeq S^8$. *Furthermore,* $(\mathrm{F}_{4(-20)})_{E_1}$ *is connected.*

(2) $(\mathrm{F}_{4(-20)})_{E_3}/\mathrm{D}_4 \simeq \mathcal{S}_+^{8,1}$. *Furthermore,* $(\mathrm{F}_{4(-20)})_{E_3}$ *is connected.*

Proof. (1) We notice that $(\mathrm{F}_{4(-20)})_{E_1,E_2-E_3} = (\mathrm{F}_{4(-20)})_{E,E_1,E_2-E_3} = (\mathrm{F}_{4(-20)})_{E_1,E_2,E_3} = \mathrm{D}_4$. From (4.6.a), we see $(\mathrm{F}_{4(-20)})_{E_1}/\mathrm{D}_4 \simeq S^8$. By Lemma 3.2(1), D_4 is connected. Obviously S^8 is connected. Hence $(\mathrm{F}_{4(-20)})_{E_1}$ is also connected.

(2) We note $\mathrm{D}_4 = (\mathrm{F}_{4(-20)})_{E_3,E_1-E_2}$. From (4.6.c), $(\mathrm{F}_{4(-20)})_{E_3}/\mathrm{D}_4 \simeq \mathcal{S}_+^{8,1}$. By Lemma 4.6, $\mathcal{S}_+^{8,1}$ is connected. Because D_4 is connected, we see that $(\mathrm{F}_{4(-20)})_{E_3}$ is also connected. $\square$

For $i \in \{1,2,3\}$, the element $\sigma_i \in \mathrm{F}_{4(-20)}$ is defined by

$$\sigma_i\left(\sum\nolimits_{j=1}^3(\xi_j E_j + F_j^1(x_j))\right) := \sum\nolimits_{j=1}^3(\xi_j E_j + \epsilon_i(j)F_j^1(x_j)).$$

Indeed, because of $\det(\sigma_i X) = \det(X)$ and $\sigma_i E = E$, applying (1.8.b), we see $\sigma_i \in \mathrm{F}_{4(-20)}$ and clearly $\sigma_i^2 = 1$. We write the notation σ instead of σ_1 for short.

The following result is proved in [16].

Proposition 4.8. (1) *The following sequence is exact:*

$$1 \to \{1, \sigma_i\} \to \mathrm{D}_4 \xrightarrow{p_i} \mathrm{SO}(F_i^1(\mathbf{O}), Q) \to 1. \tag{4.8.a}$$

(2) *The following sequence is exact:*

$$1 \to \{1, \sigma\} \to (\mathrm{F}_{4(-20)})_{E_1} \xrightarrow{\tilde{p}_1} \mathrm{SO}(\mathcal{J}^1_{L^\times(2E_1),-1}, Q) \to 1. \tag{4.8.b}$$

(3) *The following sequence is exact:*

$$1 \to \{1, \sigma_3\} \to (\mathrm{F}_{4(-20)})_{E_3} \xrightarrow{\tilde{p}_3} \mathrm{O}^0(\mathcal{J}^1_{L^\times(2E_3),-1}, Q) \to 1. \tag{4.8.c}$$

Proof. (1) It follows from Lemma 3.2(1) and (2.6.a).

(2) (cf. [19, Theorem 2.7.4]). By Lemma 4.7(1), $(\mathrm{F}_{4(-20)})_{E_1}$ is connected. Then we see that $\tilde{p}_1((\mathrm{F}_{4(-20)})_{E_1}) \subset \mathrm{SO}(\mathcal{J}^1_{L^\times(2E_1),-1}, Q)$ and the following commutative diagram:

$$\begin{array}{ccccccccc}
1 & \to & \mathrm{D}_4 & \to & (\mathrm{F}_{4(-20)})_{E_1} & \to & S^8 & \to & * \\
 & & \downarrow p_1 & & \downarrow \tilde{p}_1 & & \| & & \\
1 & \to & \mathrm{SO}(F_1^1(\mathbf{O}), Q) & \to & \mathrm{SO}(\mathcal{J}^1_{L^\times(2E_1),-1}, Q) & \to & S^8 & \to & *.
\end{array}$$

It follows from (1) and the five lemma that $\tilde{p}_1$ is onto and $\mathrm{Ker}(\tilde{p}_1) = \mathrm{Ker}(p_1) = \{1, \sigma\}$. Hence (2) follows.

(3) By Lemma 4.7(2), $(\mathrm{F}_{4(-20)})_{E_3}$ is connected. Then we see that $\tilde{p}_3((\mathrm{F}_{4(-20)})_{E_3}) \subset \mathrm{O}^0(\mathcal{J}^1_{L^\times(2E_3),-1}, Q)$ and the following commutative diagram:

$$\begin{array}{ccccccccc}
1 & \to & \mathrm{D}_4 & \to & (\mathrm{F}_{4(-20)})_{E_3} & \to & \mathcal{S}^{8,1}_+ & \to & * \\
 & & \downarrow p_3 & & \downarrow \tilde{p}_3 & & \| & & \\
1 & \to & \mathrm{SO}(F_3^1(\mathbf{O}), Q) & \to & \mathrm{O}^0(\mathcal{J}^1_{L^\times(2E_3),-1}, Q) & \to & \mathcal{S}^{8,1}_+ & \to & *.
\end{array}$$

Similarly, using (1) and the five lemma, (3) follows. □

By Lemma 4.7(1) and (4.8.b), we have $(\mathrm{F}_{4(-20)})_{E_1}$ is connected and a two-hold covering group of $\mathrm{SO}(\mathcal{J}^1_{L^\times(2E_1),-1}, Q)$, and by Lemma 4.7(2) and (4.8.c), $(\mathrm{F}_{4(-20)})_{E_3}$ is connected and a two-hold covering group of $\mathrm{O}^0(\mathcal{J}^1_{L^\times(2E_3),-1}, Q)$. So denote

$$\mathrm{Spin}(9) := (\mathrm{F}_{4(-20)})_{E_1}, \quad \mathrm{Spin}^0(8,1) := (\mathrm{F}_{4(-20)})_{E_3}.$$

Proposition 4.9. (1) *Let* $Y = (r_1 - r_2)E_1 + r_2 E \in \mathcal{J}^1$ *where* $r_1 \neq r_2$. *Then* $(\mathrm{F}_{4(-20)})_Y = \mathrm{Spin}(9)$.

(2) *Let* $Y' = (r_1 - r_2)E_3 + r_2 E \in \mathcal{J}^1$ *where* $r_1 \neq r_2$. *Then* $(\mathrm{F}_{4(-20)})_{Y'} = \mathrm{Spin}^0(8,1)$.

Proof. Since the element E is invariant under the $\mathrm{F}_{4(-20)}$-action, we have $(\mathrm{F}_{4(-20)})_Y = (\mathrm{F}_{4(-20)})_{E_1}$ and $(\mathrm{F}_{4(-20)})_{Y'} = (\mathrm{F}_{4(-20)})_{E_3}$. □

Lemma 4.10. (1) *The following equation holds.*

$$\mathcal{S}^{7,1}_{+} = Orb_{(\mathrm{F}_{4(-20)})^0_{F^1_3(1)}}(E_1 - E_2) = Orb_{(\mathrm{F}_{4(-20)})_{F^1_3(1)}}(E_1 - E_2). \tag{4.10}$$

Furthermore, $\mathcal{S}^{7,1}_{+}$ is connected.

(2) $(\mathrm{F}_{4(-20)})_{F^1_3(1)}/\mathrm{B}_3 \simeq \mathcal{S}^{7,1}_{+}$. *Furthermore, $(\mathrm{F}_{4(-20)})_{F^1_3(1)}$ is connected.*

Proof. (1) Note $\mathcal{S}^{7,1}_{+} = \mathcal{S}^{8,1}_{+} \cap \mathcal{S}^{7,1}$. Let $X \in \mathcal{S}^{7,1}_{+}$ and $g \in (\mathrm{F}_{4(-20)})_{F^1_3(1)}$. Because of $X \in \mathcal{S}^{8,1}_{+}$ and (4.6.c), $gX \in \mathcal{S}^{8,1}_{+}$. Next, $0 = (F^1_3(1)|X) = (gF^1_3(1)|gX) = (F^1_3(1)|gX)$. Thus $gX \in \mathcal{S}^{7,1}_{+}$ and so $(\mathrm{F}_{4(-20)})_{F^1_3(1)}$ acts on $\mathcal{S}^{7,1}_{+}$. Especially, $(\mathrm{F}_{4(-20)})^0_{F^1_3(1)}$ acts on $\mathcal{S}^{7,1}_{+}$.

Next, we will show transitivity. Fix $X \in \mathcal{S}^{7,1}_{+}$. X is expressed by $X = \xi(E_1 - E_2) + F^1_3(x)$ where $\xi > 0$, $x \in \mathrm{Im}\mathbf{O}$ and $\xi^2 - \mathrm{n}(x) = 1$. Using (2.4.a), $gx = \sqrt{\mathrm{n}(x)}e_1$ for some $g \in \mathrm{G}_2$. Then $\varphi_0(g,g,g) \in \mathrm{G}_2 \subset \mathrm{B}_3 \subset (\mathrm{F}_{4(-20)})^0_{F^1_3(1)}$ and $\varphi_0(g,g,g)X = \xi(E_1 - E_2) + F^1_3(\sqrt{\mathrm{n}(x)}e_1)$. Put $t_0 = 4^{-1}\log\left((\xi + \sqrt{\mathrm{n}(x)})/(\xi - \sqrt{\mathrm{n}(x)})\right) \in \mathbb{R}$. Because of $e_1 \in S^6$ and Lemma 3.10(2), we see $\exp(t_0\tilde{A}^1_3(e_1)) \in (\mathrm{F}_{4(-20)})^0_{F^1_3(1)}$ and because of $\xi^2 - \mathrm{n}(x) = 1$ and (3.10.b), we calculate $\exp\left(t_0\tilde{A}^1_3(e_1)\right)\varphi_0(g,g,g)X = E_1 - E_2$. Thus $\mathcal{S}^{7,1}_{+} = Orb_{(\mathrm{F}_{4(-20)})^0_{E_3}}(E_1 - E_2) = Orb_{(\mathrm{F}_{4(-20)})_{E_3}}(E_1 - E_2)$. Because $\mathcal{S}^{7,1}_{+}$ is an orbit of one element $E_1 - E_2$ under the action of a connected group $(\mathrm{F}_{4(-20)})^0_{E_3}$, $\mathcal{S}^{7,1}_{+}$ is connected. Hence (1) follows.

(2) Note $(\mathrm{F}_{4(-20)})_{E_1-E_2,E_3} = (\mathrm{F}_{4(-20)})_{E,E_1-E_2,E_3} = (\mathrm{F}_{4(-20)})_{E_1,E_2,E_3}$. By (3.1.a) and (3.1.b), $(\mathrm{F}_{4(-20)})_{F^1_3(1),E_1-E_2} = (\mathrm{F}_{4(-20)})_{F^1_3(1),E_1-E_2,E_3} = (\mathrm{F}_{4(-20)})_{F^1_3(1),E_1,E_2,E_3} = \mathrm{B}_3$. From (1), we see $(\mathrm{F}_{4(-20)})_{F^1_3(1)}/\mathrm{B}_3 \simeq \mathcal{S}^{7,1}_{+}$. By Lemma 3.2(2), B_3 is connected and by (1), $\mathcal{S}^{7,1}_{+}$ are connected. Hence $(\mathrm{F}_{4(-20)})_{F^1_3(1)}$ is also connected. $\square$

Proposition 4.11. (1) *The following sequence is exact:*

$$1 \to \{1, \sigma_3\} \to \mathrm{B}_3 \xrightarrow{q} \mathrm{SO}(F^1_3(\mathrm{Im}\mathbf{O}), Q) \to 1. \tag{4.11.a}$$

(2) *The following sequence is exact:*

$$1 \to \{1, \sigma_3\} \to (\mathrm{F}_{4(-20)})_{F^1_3(1)} \xrightarrow{\tilde{q}} \mathrm{O}^0(\mathcal{J}^1_{7,1}, Q) \to 1. \tag{4.11.b}$$

Proof. (1) It follows from Lemma 3.2(2) and (2.6.b).

(2) By Lemma 4.10(2), $(\mathrm{F}_{4(-20)})_{F^1_3(1)}$ is connected. Then we see that $\tilde{q}((\mathrm{F}_{4(-20)})_{F^1_3(1)}) \subset \mathrm{O}^0(\mathcal{J}^1_{7,1}, Q)$ and the following commutative diagram:

$$\begin{array}{ccccccccc}
1 & \to & \mathrm{B}_3 & \to & (\mathrm{F}_{4(-20)})_{F^1_3(1)} & \to & \mathcal{S}^{7,1}_{+} & \to & * \\
 & & \downarrow q & & \downarrow \tilde{q} & & \| & & \\
1 & \to & \mathrm{SO}(F^1_3(\mathrm{Im}\mathbf{O}), Q) & \to & \mathrm{O}^0(\mathcal{J}^1_{7,1}, Q) & \to & \mathcal{S}^{7,1}_{+} & \to & *.
\end{array}$$

It follows from (1) and the five lemma that $\tilde{q}$ is onto and $\mathrm{Ker}(\tilde{q}) = \mathrm{Ker}(q) = \{1, \sigma_3\}$. Hence the assertion follows. $\square$

By Lemma 4.10(2) and (4.11.b), we have $(\mathrm{F}_{4(-20)})_{F^1_3(1)}$ is connected and a two-hold covering group of $\mathrm{O}^0(\mathcal{J}^1_{7,1}, Q)$. So denote

$$\mathrm{Spin}^0(7,1) := (\mathrm{F}_{4(-20)})_{F^1_3(1)}.$$

Proposition 4.12. *Let $Y = rE_3 + p(E - E_3) + qF^1_3(1) \in \mathcal{J}^1$ with $q \neq 0$. Then $(\mathrm{F}_{4(-20)})_Y = \mathrm{Spin}^0(7,1)$.*

Proof. From (3.1.a), we see $\mathrm{Spin}^0(7,1) = (\mathrm{F}_{4(-20)})_{E_3,F^1_3(1)} \subset (\mathrm{F}_{4(-20)})_Y$. Conversely, take $g \in (\mathrm{F}_{4(-20)})_Y$. Because of $(rE - Y)^{\times 2} = ((p-r)^2 + q^2)E_3$ and $\mathrm{tr}((rE-Y)^{\times 2}) = (p-r)^2 + q^2 \neq 0$, we see $E_{Y,r} \in \mathcal{J}^1$ is well-defined and $E_{Y,r} = E_3$. By (1.10)(iii), $gE_3 = gE_{Y,r} = E_{gY,r} = E_{Y,r} = E_3$. Then $gF^1_3(1) = g(q^{-1}(Y - (r-p)E_3 - p(E - E_3))) = F^1_3(1)$. Thus $g \in \mathrm{Spin}^0(7,1)$ and so $(\mathrm{F}_{4(-20)})_Y \subset \mathrm{Spin}^0(7,1)$. Hence $(\mathrm{F}_{4(-20)})_Y = \mathrm{Spin}^0(7,1)$. □

For $i \in \{1,2,3\}$, the involutive automorphism $\tilde{\sigma}_i$ of $\mathrm{F}_{4(-20)}$ is defined by $\tilde{\sigma}_i(g) := \sigma_i g \sigma_i$ for $g \in \mathrm{F}_{4(-20)}$ and the subgroup K of $\mathrm{F}_{4(-20)}$ by

$$K := (\mathrm{F}_{4(-20)})^{\tilde{\sigma}} = \{g \in \mathrm{F}_{4(-20)} |\ \sigma g = g\sigma\}.$$

Lemma 4.13. *Let $i, j \in \{1,2,3\}$ and indexes $i, i+1, i+2, i+j$ be counted modulo 3.*

(1) *The following expressions hold.*

(4.13.a)
$$\begin{cases} \text{(i)} \quad \mathcal{J}^1_{\sigma_i} &= \{(\sum_{j=1}^{3}\xi_j E_j) + F^1_i(x) | \xi_j \in \mathbb{R}, x \in \mathbf{O}\} \\ &= \{X \in \mathcal{J}^1 \mid 4E_i \times (E_i \times X) = X\} \oplus \mathbb{R}E_i, \\ \text{(ii)} \quad \mathcal{J}^1_{\sigma_i,-1} &= \{\sum_{j=1}^{2} F^1_{i+j}(x_{i+j}) | x_{i+j} \in \mathbf{O}\} \\ &= \{X \in \mathcal{J}^1 \mid E_i \times X = 0,\ (E_i|X) = 0\}. \end{cases}$$

(2) *$(\mathrm{F}_{4(-20)})^{\tilde{\sigma}_i}$ invariants the linear subspaces $\mathcal{J}^1_{\sigma_i}$ and $\mathcal{J}^1_{\sigma_i,-1}$ of $\mathcal{J}^1$.*

(3) *Let $g \in (\mathrm{F}_{4(-20)})^{\tilde{\sigma}_i}$. Then*

(4.13.b)
$$gE_i = E_i + \xi_{i+1}E_{i+1} + \xi_{i+2}E_{i+2} + F^1_i(x)$$

for some $\xi_{i+1}, \xi_{i+2} \in \mathbb{R}$ and $x \in \mathbf{O}$.

Proof. (1) Using the definition of σ_i and Lemma 1.6, it follows from direct calculations.

(2) It follows from $\sigma_i g = g\sigma_i$ for all $g \in (\mathrm{F}_{4(-20)})^{\tilde{\sigma}_i}$.

(3) (cf. [19, Theorem 2.9.1]). Now $F^1_{i+1}(1),\ F^1_{i+2}(1) \in \mathcal{J}^1_{\sigma_i,-1}$. By (2), $gF^1_{i+1}(1),\ gF^1_{i+2}(1) \in \mathcal{J}^1_{\sigma_i,-1}$ and from (4.13.a), we can write $gF^1_{i+1}(1) = \sum_{j=1}^{2} F^1_{i+j}(x_{i+j})$ and $gF^1_{i+2}(1) = \sum_{j=1}^{2} F^1_{i+j}(y_{i+j})$ for some $x_{i+j}, y_{i+j} \in \mathbf{O}$. By (1.6.a), $E_{i+1} = -\epsilon(i+1)(F^1_{i+1}(1))^{\times 2}$ and $E_{i+2} = -\epsilon(i+2)(F^1_{i+2}(1))^{\times 2}$, so that $gE_{i+1} = -\epsilon(i+1)(gF^1_{i+1}(1))^{\times 2}$ and $gE_{i+2} = -\epsilon(i+2)(gF^1_{i+2}(1))^{\times 2}$. From (1.6.d), we see that $gE_{i+1} = -\epsilon(i+1)(\sum_{j=1}^{2} F^1_{i+j}(x_{i+j}))^{\times 2} = (\sum_{j=1}^{2} \xi_{i+j}E_{i+j}) + F^1_i(u)$ and $gE_{i+2} = -\epsilon(i+2)(\sum_{j=1}^{2} F^1_{i+j}(y_{i+j}))^{\times 2} = (\sum_{j=1}^{2} \eta_{i+j}E_{i+j}) + F^1_i(v)$ for some $\xi_{i+j}, \eta_{i+j} \in$

$\mathbb{R}$ and $u, v \in \mathbf{O}$. Thus $gE_i = g(E - E_{i+1} - E_{i+2}) = E - \sum_{j=1}^{2}(\xi_{i+j} + \eta_{i+j})E_{i+j} - F_i^1(u+v)$. Hence (3) follows. $\square$

The following result is shown in [16].

Proposition 4.14. *Let $i \in \{1, 2, 3\}$. Then the following equation hold.*

(4.14.a) $$(\mathrm{F}_{4(-20)})^{\tilde{\sigma}_i} = (\mathrm{F}_{4(-20)})_{E_i}.$$

Especially,

(4.14.b) $$K = (\mathrm{F}_{4(-20)})_{E_1} = \mathrm{Spin}(9),$$

(4.14.c) $$(\mathrm{F}_{4(-20)})^{\tilde{\sigma}_2} = (\mathrm{F}_{4(-20)})_{E_2} \cong \mathrm{Spin}^0(8,1).$$

Proof. (cf. [19, Theorem 2.9.1]). Fix $g \in (\mathrm{F}_{4(-20)})_{E_i}$. For all $X \in \mathcal{J}^1$, X can be expressed by $X = X_{\sigma_i} + X_{-\sigma_i}$ for some $X_{\sigma_i} \in \mathcal{J}^1_{\sigma_i}$ and $X_{-\sigma_i} \in \mathcal{J}^1_{\sigma_i,-1}$. From (4.13.a), we see $gX_{\sigma_i} \in \mathcal{J}^1_{\sigma_i}$ and $gX_{-\sigma_i} \in \mathcal{J}^1_{\sigma_i,-1}$. Then $g\sigma_i X = gX_{\sigma_i} - gX_{-\sigma_i} = \sigma_i gX$. Hence $g \in (\mathrm{F}_{4(-20)})^{\tilde{\sigma}_i}$ and so $(\mathrm{F}_{4(-20)})_{E_i} \subset (\mathrm{F}_{4(-20)})^{\tilde{\sigma}_i}$.

Conversely, take $g \in (\mathrm{F}_{4(-20)})^{\tilde{\sigma}_i}$. Let index $i+j$ be counted modulo 3. By (4.13.b), $gE_i = E_i + \xi_{i+1}E_{i+1} + \xi_{i+2}E_{i+2} + F_i^1(x)$ for some $\xi_{i+1}, \xi_{i+2} \in \mathbb{R}$ and $x \in \mathbf{O}$. Because of $(gE_i)^{\times 2} = g(E_i^{\times 2}) = 0$ and (1.6.d), we see $0 = ((gE_i)^{\times 2})_{E_{i+j}} = \xi_{i+j}$ and $0 = ((gE_i)^{\times 2})_{F_i^1} = -x$. Then $gE_i = E_i$. Thus $g \in (\mathrm{F}_{4(-20)})_{E_i}$ and so $(\mathrm{F}_{4(-20)})^{\tilde{\sigma}_i} \subset (\mathrm{F}_{4(-20)})_{E_i}$. Hence (4.14.a) follows. $\square$

5. The exceptional hyperbolic planes and the exceptional null cones.

In this section, we define the exceptional hyperbolic planes and the exceptional null cones, and we will show Proposition 0.1. Denote

$$\mathcal{H} := \{X \in \mathcal{J}^1 |\ X^{\times 2} = 0,\ \mathrm{tr}(X) = 1\},$$
$$\mathcal{H}(\mathbf{O}) := \{X \in \mathcal{H} |\ (X|E_1) \geq 1\},$$
$$\mathcal{H}'(\mathbf{O}) := \{X \in \mathcal{H} |\ (X|E_1) \leq 0\}.$$

Then $\mathcal{H}(\mathbf{O})$ and $\mathcal{H}'(\mathbf{O})$ are called the *hyperbolic planes* of $\mathbf{O}$ or the *exceptional hyperbolic planes*. The cone $\mathcal{N}$ in $\mathcal{J}^1$ is defined by

$$\mathcal{N} := \{X \in \mathcal{J}^1 \mid \mathrm{tr}(X) = \mathrm{tr}(X^{\times 2}) = \det(X) = 0\}.$$

We recall $(X^{\times 2})^{\times 2} = \det(X)X$ and observe that the cone $\mathcal{N}$ contains the following cones:

$$\mathcal{N}_1(\mathbf{O}) := \{X \in \mathcal{J}^1 |\ X^{\times 2} = 0,\ \mathrm{tr}(X) = 0,\ X \neq 0\},$$
$$\mathcal{N}_1^+(\mathbf{O}) := \{X \in \mathcal{J}^1 |\ X^{\times 2} = 0,\ \mathrm{tr}(X) = 0,\ (X|E_1) > 0\},$$
$$\mathcal{N}_1^-(\mathbf{O}) := \{X \in \mathcal{J}^1 |\ X^{\times 2} = 0,\ \mathrm{tr}(X) = 0,\ (X|E_1) < 0\},$$

and

$$\mathcal{N}_2(\mathbf{O}) := \{X \in \mathcal{J}^1 |\ \mathrm{tr}(X) = \mathrm{tr}(X^{\times 2}) = \det(X) = 0, X^{\times 2} \neq 0\}.$$

Then $\mathcal{N}_1^{\pm}(\mathbf{O})$ and $\mathcal{N}_2(\mathbf{O})$ are called the *exceptional null cones*. We write $\mathcal{N}_0(\mathbf{O})$ for the trivial space $\{0\}$ in $\mathcal{J}^1$.

Lemma 5.1. *The group* $\mathrm{F}_{4(-20)}$ *acts on* $\mathcal{H}$*,* $\mathcal{N}_1(\mathbf{O})$ *and* $\mathcal{N}_2(\mathbf{O})$ *and*

$$\mathcal{H} = \mathcal{H}(\mathbf{O}) \coprod \mathcal{H}'(\mathbf{O}), \tag{5.1.a}$$

$$\mathcal{N}_1(\mathbf{O}) = \mathcal{N}_1^{+}(\mathbf{O}) \coprod \mathcal{N}_1^{-}(\mathbf{O}). \tag{5.1.b}$$

Proof. From the definitions of $\mathcal{H}, \mathcal{N}_1(\mathbf{O})$ and $\mathcal{N}_2(\mathbf{O})$, we see that $\mathrm{F}_{4(-20)}$ acts on these spaces. We show (5.1.a). $\mathcal{H}(\mathbf{O}) \cap \mathcal{H}'(\mathbf{O}) = \emptyset$ is obvious. Fix $X = \sum_{i=1}^{3}(\xi_i E_i + F_i^1(x_i)) \in \mathcal{H}$ where $\xi_i \in \mathbb{R}$ and $x_i \in \mathbf{O}$. By (1.6.d), $0 = (X^{\times 2})_{E_2} = \xi_3\xi_1 + (x_2|x_2)$ and $0 = (X^{\times 2})_{E_3} = \xi_1\xi_2 + (x_3|x_3)$. Then $\xi_1(\xi_2+\xi_3) = -(x_2|x_2) - (x_3|x_3) \leq 0$, so that $(X)_{E_1} = \xi_1 \leq 0$ or $\xi_2+\xi_3 \leq 0$. If $\xi_2 + \xi_3 \leq 0$, then $(X|E_1) = \xi_1 = \mathrm{tr}(X) - (\xi_2+\xi_3) = 1 - (\xi_2+\xi_3) \geq 1$. Hence (5.1.a) follows.

Next, we show (5.1.b). $\mathcal{N}_1^{+}(\mathbf{O}) \cap \mathcal{N}_1^{-}(\mathbf{O}) = \emptyset$ is obvious. Suppose that $X \in \mathcal{N}_1(\mathbf{O})$ and $\xi_1 = (X|E_1) = 0$. We can write $X = \xi E_2 - \xi E_3 + \sum_{i=1}^{3} F_i^1(x_i)$ where $\xi \in \mathbb{R}$ and $x_i \in \mathbf{O}$. Then $0 = (X^{\times 2})_{E_1} = -\xi^2 - (x_1|x_1)$ and $0 = (X^{\times 2})_{E_i} = (x_i|x_i)$ $(i = 2, 3)$, and therefore $\xi = 0$ and $x_i = 0$ for all $i \in \{1, 2, 3\}$ iff $X = 0$. It contradicts with $X \neq 0$. Hence (5.1.b) follows. □

Denote $\mathcal{J}^1(2;\mathbf{O}) := \{\xi_1 E_1 + \xi_2 E_2 + F_3^1(x) |\ \xi_i \in \mathbb{R},\ x \in \mathbf{O}\}$.

Lemma 5.2. (1) *For any* $X \in \mathcal{J}^1$*, there exists* $g \in K$ *such that* $(gX)_{F_1} = 0$ *and* $(gX|E_1) = (X|E_1)$*.*

(2) *Assume* $X \in \mathcal{J}^1$ *satisfies* $X^{\times 2} = 0$*. Then there exists* $g \in K$ *such that* $gX \in \mathcal{J}^1(2;\mathbf{O})$ *and* $(gX|E_1) = (X|E_1)$*.*

(3) *For any* $X \in \mathcal{H}$*, there exists* $g \in K$ *such that* $gX = 2^{-1}(E - E_3) + 2^{-1}W$ *where* $W \in \mathcal{S}^{8,1}$ *and* $(gX|E_1) = (X|E_1)$*.*

(4) *For any* $X \in \mathcal{N}_1(\mathbf{O})$*, there exists* $g \in K$ *such that* $gX \in \mathcal{N}^{8,1}$ *and* $(gX|E_1) = (X|E_1)$*.*

Proof. From (4.14.b), we note $(kX|E_1) = (kX|kE_1) = (X|E_1)$ for all $k \in K$. Thus it is enough to show the conditions in addition to the condition $(gX|E_1) = (X|E_1)$.

(1) Set $X = \sum_{i=1}^{3}(\xi_i E_i + F_i^1(x_i))$. We consider the decomposition

$$\begin{aligned} X =& \xi_1 E_1 + 2^{-1}(\xi_2+\xi_3)(E - E_1) + \big(2^{-1}(\xi_2-\xi_3)(E_2 - E_3) + F_1^1(x_1)\big) \\ &+ (F_2^1(x_2) + F_3^1(x_3)). \end{aligned}$$

Put $X_0 = \xi_1 E_1 + 2^{-1}(\xi_2+\xi_3)(E - E_1)$, $Y = 2^{-1}(\xi_2-\xi_3)(E_2-E_3) + F_1^1(x_1)$ and $X_{-\sigma} = F_2^1(x_2) + F_3^1(x_3)$. By (4.1.c), $Y \in \mathcal{J}^1_{L^{\times}(2E_1),-1}$ and by (4.13.a), $X_{-\sigma} \in \mathcal{J}^1_{\sigma,-1}$. Fix $k_0 \in K$. From (4.14.b) and Lemma 4.13(2), we see $kX_0 = X_0$ and $k_0 X_{-\sigma} \in \mathcal{J}^1_{\sigma,-1}$ so that $(k_0 X_0)_{F_1^1} = 0$ and $(k_0 X_{-\sigma})_{F_1^1} = 0$ (see (4.13.a)). Now, we can write $Y = rW$ for some $r \in \mathbb{R}$ and $W \in S^8$. By (4.6.a), there exists $g \in K$ such that

$gW = r(E_2 - E_3)$. Thus $(gW)_{F_1^1} = 0$ and so $(gX)_{F_1^1} = (gX_0)_{F_1^1} + (gX_{-\sigma})_{F_1^1} + r(gW)_{F_1^1} = 0$.

(2) By (1), there exists $g \in K$ such that $gX = (\sum_{i=1}^{3} r_i E_i) + (\sum_{i=2}^{3} F_i^1(y_i))$ where $r_i \in \mathbb{R}$, $y_i \in \mathbf{O}$ and $(X|E_1) = (gX|E_1) = r_1$. From the assumption, we see $0 = g(X^{\times 2}) = (gX)^{\times 2}$. By (1.6.d), $0 = ((gX)^{\times 2})_{E_1} = r_2 r_3$. Then we have the following two cases: (i) $r_3 = 0$ or (ii) $r_2 = 0$.

Case (i) $r_3 = 0$. Then $0 = ((gX)^{\times 2})_{E_2} = (y_2|y_2)$ and therefore $y_2 = 0$. Thus $gX = r_1E_1 + r_2E_2 + F_3^1(y_3) \in \mathcal{J}^1(2;\mathbf{O})$.

Case (ii) $r_2 = 0$. Then $0 = ((gX)^{\times 2})_{E_3} = (y_3|y_3)$ and therefore $y_3 = 0$. From Lemma 3.10, we see $\exp(2^{-1}\pi \tilde{A}_1^1(1)) \in (\mathrm{F}_{4(-20)})_{E_1} = K$ and $\exp((2^{-1}\pi)\tilde{A}_1^1(1))gX = r_1E_1 + r_3E_2 + F_3^1(\overline{y_2}) \in \mathcal{J}^1(2;\mathbf{O})$. Hence (2) follows.

(3) By (2), there exists $g \in (\mathrm{F}_{4(-20)})_{E_1}$ such that $gX = \xi_1E_1 + \xi_2E_2 + F_3^1(x) \in \mathcal{J}^1(2;\mathbf{O})$ with $1 = \mathrm{tr}(X) = \mathrm{tr}(gX) = \xi_1 + \xi_2$. Put $Y = 2^{-1}(\xi_1 - \xi_2)(E_1 - E_2) + F_3^1(x)$. Then $gX = 2^{-1}(E - E_3) + Y$ and $Y \in \mathcal{J}^1_{L^\times(2E_3),-1}$. Because of $(gX)^{\times 2} = g(X^{\times 2}) = 0$, using (1.6.d), we see $0 = (g(X^{\times 2}))_{E_3} = ((gX)^{\times 2})_{E_3} = \xi_1\xi_2 + (x|x)$. Then by (4.1.d), $Q(Y) = 4^{-1}(\xi_1 - \xi_2)^2 - (x|x) = 4^{-1}(\xi_1 + \xi_2)^2 - (\xi_1\xi_2 + (x|x)) = 4^{-1}$. Thus $Y \in 2^{-1}\mathcal{S}^{8,1}$ and so (3) follows.

(4) By (2), there exists $g \in K$ such that $(0 \neq) gX = \xi_1E_1 + \xi_2E_2 + F_3^1(x)$. Because of (4.1.a), (4.1.c) and $\xi_1 + \xi_2 = \mathrm{tr}(gX) = \mathrm{tr}(X) = 0$, we see that $Q(gX) = 0$ and $gX = \xi_1(E_1 - E_2) + F_3^1(x) \in \mathcal{J}^1_{L^\times(2E_3),-1}$. Hence $gX \in \mathcal{N}^{8,1}$. □

Proof of (0.1.a) and (0.1.b).

We show these by using Lemma 4.4. By Lemma 5.1, $\mathrm{F}_{4(-20)}$ acts on $\mathcal{H}$. We consider that $X = \mathcal{H}$, $G = \mathrm{F}_{4(-20)}$, $(X_1, X_2) = (\mathcal{H}(\mathbf{O}), \mathcal{H}'(\mathbf{O}))$, $(v_1, v_2) = (E_1, E_2)$ in Lemma 4.4. At first, the condition (i) follows from (5.1.a). At second, the condition (ii) follows from direct calculations. At third, the condition (iii) and $Orb_{\mathrm{F}_{4(-20)}}(E_2) = Orb_{\mathrm{F}_{4(-20)}}(E_3)$ follow from (3.11.a). At last, we show the condition (iv). Take $X \in \mathcal{H} = \mathcal{H}(\mathbf{O}) \coprod \mathcal{H}'(\mathbf{O})$. By Lemma 5.2(3), there exists $g_0 \in K$ such that

$$g_0X = 2^{-1}(E - E_3) + 2^{-1}W$$

where $W \in \mathcal{S}^{8,1}$ and $(g_0X|E_1) = (X|E_1)$.

Case $X \in \mathcal{H}(\mathbf{O})$. Because of $(g_0X|E_1) = (X|E_1) \geq 1$, we see $(W|E_1) = 2(g_0X|E_1) - (E - E_3|E_1) = 2(g_0X|E_1) - 1 > 0$ so that $W \in \mathcal{S}^{8,1}_+$. By (4.6.c)(i), there exists $g_1 \in (\mathrm{F}_{4(-20)})_{E_3}$ such that $g_1W = E_1 - E_2$ and it is clear that

$$g_1g_0X = 2^{-1}(E_1 + E_2) + 2^{-1}(E_1 - E_2) = E_1.$$

Thus $X \in Orb_{\mathrm{F}_{4(-20)}}(E_1)$ and so $\mathcal{H}(\mathbf{O}) \subset Orb_{\mathrm{F}_{4(-20)}}(E_1)$.

Case $X \in \mathcal{H}'(\mathbf{O})$. Because of $(g_0X|E_1) = (X|E_1) \leq 0$, $(W|E_1) = 2(g_0X|E_1) - 1 < 0$. Then $W \in \mathcal{S}^{8,1}_-$. By (4.6.c)(ii), there exists $g_1 \in$

$(\mathrm{F}_{4(-20)})_{E_3}$ such that $g_1'W = -E_1 + E_2$ and it is clear that

$$g_1'g_0X = 2^{-1}(E_1 + E_2) + 2^{-1}(-E_1 + E_2) = E_2.$$

Thus $X \in Orb_{\mathrm{F}_{4(-20)}}(E_2)$ and so $\mathcal{H}(\mathbf{O}) \subset Orb_{\mathrm{F}_{4(-20)}}(E_2)$.

Therefore the condition (iv) follows. Hence (0.1.a) and (0.1.b) follow from Lemma 4.4. □

Lemma 5.3.

$$\mathcal{N} = \{0\} \coprod \mathcal{N}_1^+(\mathbf{O}) \coprod \mathcal{N}_1^-(\mathbf{O}) \coprod \mathcal{N}_2(\mathbf{O}). \tag{5.3}$$

Proof. Let $X \in \mathcal{N}$. By (1.3.b)(iv), $(X^{\times 2})^{\times 2} = \det(X)X = 0$. Then we have the following three cases: (i) $X = 0$, (ii) $X \neq 0$, $X^{\times 2} = 0$ or (iii) $X^{\times 2} \neq 0$. It implies $X \in \{0\} \coprod \mathcal{N}_1(\mathbf{O}) \coprod \mathcal{N}_2(\mathbf{O})$. Hence (5.3) follows from (5.1.b). □

Proof of (0.1.c) and (0.1.d).

We show these by using Lemma 4.4. By Lemma 5.1, $\mathrm{F}_{4(-20)}$ acts on $\mathcal{N}_1(\mathbf{O})$. We consider that $X = \mathcal{N}_1(\mathbf{O})$, $G = \mathrm{F}_{4(-20)}$, $(X_1, X_2) = (\mathcal{N}_1^+(\mathbf{O}), \mathcal{N}_1^-(\mathbf{O}))$, $(v_1, v_2) = (P^+, P^-)$ in Lemma 4.4. At first, the condition (i) follows from (5.1.b). At second, the condition (ii) follows from direct calculations. At third, the condition (iii) follows from (3.12.c). At last, we show the condition (iv). Take $X \in \mathcal{N}_1(\mathbf{O}) = \mathcal{N}_1^+(\mathbf{O}) \coprod \mathcal{N}_1^-(\mathbf{O})$. By Lemma 5.2(4), there exists $g_0 \in K$ such that $g_0X \in \mathcal{N}^{8,1}$ and $(g_0X|E_1) = (X|E_1)$.

Case $X \in \mathcal{N}_1^+(\mathbf{O})$. Because of $(g_0X|E_1) = (X|E_1) > 0$, we see $g_0X \in \mathcal{N}_+^{8,1}$. By (4.6.d)(i), there exists $g_1 \in (\mathrm{F}_{4(-20)})_{E_3}$ such that $g_1g_0X = P^+$. Thus $X \in Orb_{\mathrm{F}_{4(-20)}}(P^+)$ and so $\mathcal{N}_1^+(\mathbf{O}) \subset Orb_{\mathrm{F}_{4(-20)}}(P^+)$.

Case $X \in \mathcal{N}_1^-(\mathbf{O})$. Because of $(g_0X|E_1) = (X|E_1) < 0$, we see $g_0X \in \mathcal{N}_-^{8,1}$. By (4.6.d)(ii), there exists $g_1 \in (\mathrm{F}_{4(-20)})_{E_3}$ such that $g_1g_0X = P^-$. Thus $X \in Orb_{\mathrm{F}_{4(-20)}}(P^+)$ and so $\mathcal{N}_1^-(\mathbf{O}) \subset Orb_{\mathrm{F}_{4(-20)}}(P^+)$.

Therefore the condition (iv) follows. Hence (0.1.c) and (0.1.d) follows from Lemma 4.4. □

For $t \in \mathbb{R}$, denote

$$\alpha_{1,2}(t) := \exp(t(\tilde{A}_1^1(-1) + \tilde{A}_2^1(1))) \in \mathrm{F}_{4(-20)}.$$

Lemma 5.4. *Let $X_0 = rP^- + Q^+(x)$ where $r \in \mathbb{R}$ and $x \in \mathbf{O}$. Then*

$$\alpha_{1,2}(t)X_0 = (r - 2t\mathrm{Re}(x))P^- + Q^+(x). \tag{5.4}$$

Especially, $\alpha_{1,2}(t) \in (\mathrm{F}_{4(-20)})_{P^-}$.

Proof. Using (3.6), we see that $(\tilde{A}_1^1(-1)+\tilde{A}_2^1(1))P^- = 0$ and $(\tilde{A}_1^1(-1)+\tilde{A}_2^1(1))Q^+(x) = -2\mathrm{Re}(x)P^-$. Thus $\exp(t(\tilde{A}_1^1(-1) + \tilde{A}_2^1(1))P^- = P^-$ and $\exp(t(\tilde{A}_1^1(-1) + \tilde{A}_2^1(1))Q^+(x) = Q^+(x) - 2t\mathrm{Re}(x)P^-$. Hence the result follows. □

Lemma 5.5. *Let $X \in \mathcal{R}^-$. Then there exists $g \in (\mathrm{F}_{4(-20)})_{P^-}$ such that*

$$gX = Q^+(1).$$

Proof. By (3.12.b), X can be expressed by

$$X = rP^- + Q^+(x) \quad \text{for some } r \in \mathbb{R} \text{ and } x \in S^7.$$

(Step 1) We show the following assertion: if $x \in \mathrm{Im}\mathbf{O}$, then there exists $g_0 \in (\mathrm{F}_{4(-20)})_{P^-}$ such that $g_0 X = rP^- + Q^+(x')$ where $x' \in S^7$ and $\mathrm{Re}(x') \neq 0$. Suppose that $x \in S^6$. By (2.4.a), there exists $g_1 \in \mathrm{G}_2$ such that $g_1 x = e_1$. Put $\alpha_1 = \varphi_0(g_1, g_1, g_1) \in \mathrm{G}_2 \subset (\mathrm{F}_{4(-20)})_{P^-}$. By (3.3), $\alpha_1 X = rP^- + Q^+(g_0 x) = rP^- + Q^+(e_1)$. Because of $e_2, e_3 \in S^6$ and Lemma 2.4(3), we can set $\alpha_1 = (L_{e_3,e_2}, R_{e_3,e_2}, T_{e_3,e_2}) \in \mathrm{B}_3 \subset (\mathrm{F}_{4(-20)})_{P^-}$. By (3.3), $\alpha_2\alpha_1 X = rP^- + Q^+(e_3(e_2e_1)) = rP^- + Q^+(1)$. Hence the assertion of (Step 1) follows.

(Step 2) We may assume $X = rP^- + Q^+(x)$ where $r \in \mathbb{R}$, $x \in S^7$ and $\mathrm{Re}(x) \neq 0$ by (Step 1). From (5.4), we see that $\alpha_{1,2}(t) \in (\mathrm{F}_{4(-20)})_{P^-}$ and $\alpha_{1,2}\left(r/(2\mathrm{Re}(x))\right) X = Q^+(x)$.

(Step 3) We may assume $X = Q^+(x)$ where $x \in S^7$ by (Step 2). Then x can be expressed by $x = \cos\theta + a \sin\theta$ for some $\theta \in \mathbb{R}$ and $a \in S^6$. By (2.4.a), there exists $g_1 \in \mathrm{G}_2$ such that $g_1 a = e_1$. Letting $\alpha_1 = \varphi_0(g_1, g_1, g_1) \in \mathrm{G}_2 \subset (\mathrm{F}_{4(-20)})_{P^-}$,

$$\alpha_1 X = Q^+(g_1 x) = Q^+(\cos\theta + e_1 \sin\theta).$$

Letting $\alpha_2 = (L_{e_1,e_2}, R_{e_1,e_2}, T_{e_1,e_2}) \in \mathrm{B}_3 \subset (\mathrm{F}_{4(-20)})_{P^-}$,

$$\alpha_2\alpha_1 X = Q^+\left(e_1(e_2(\cos\theta + e_1 \sin\theta))\right) = Q^+(e_3 \cos\theta + e_2 \sin\theta).$$

Again, there exists $g_2 \in \mathrm{G}_2$ such that $g_2(e_3 \cos\theta + e_2 \sin\theta) = e_1$. Letting $\alpha_3 = \varphi_0(g_2, g_2, g_2) \in \mathrm{G}_2 \subset (\mathrm{F}_{4(-20)})_{P^-}$,

$$\alpha_3\alpha_2\alpha_1 X = Q^+(e_1).$$

Last, letting $\alpha_4 = \varphi_0(L_{e_3,e_2}, R_{e_3,e_2}, T_{e_3,e_2}) \in \mathrm{B}_3 \subset (\mathrm{F}_{4(-20)})_{P^-}$,

$$\alpha_4\alpha_3\alpha_2\alpha_1 X = Q^+(e_3(e_2e_1)) = Q^+(1).$$

Hence the result follows. □

Lemma 5.6. *Let $X \in \mathcal{N}_2(\mathbf{O})$. Then $X^{\times 2} \in \mathcal{N}_1^-(\mathbf{O})$.*

Proof. Obviously $\mathrm{tr}(X^{\times 2}) = 0$ and from (1.3.b)(iv), we see $(X^{\times 2})^{\times 2} = \det(X)X = 0$ so that $X^{\times 2} \in \mathcal{N}_1(\mathbf{O})$. By (5.1.b), $X^{\times 2} \in \mathcal{N}_1^+(\mathbf{O})$ or $X^{\times 2} \in \mathcal{N}_1^-(\mathbf{O})$. Suppose $X^{\times 2} \in \mathcal{N}_1^+(\mathbf{O})$. By (0.1.c), there exists $g \in \mathrm{F}_{4(-20)}$ such that $P^+ = g(X^{\times 2}) = (gX)^{\times 2}$ and $\mathrm{tr}(gX) = \mathrm{tr}(X) = 0$. Then $gX \in \mathcal{R}^+$. Thus it contradicts with (3.12.a). Hence the result follows. □

Proof of (0.1.e).

By Lemma 5.1, $\mathrm{F}_{4(-20)}$ acts on $N_2(\mathbf{O})$. We show transitivity. Fix $X \in \mathcal{N}_2(\mathbf{O})$. By Lemma 5.5, $X^{\times 2} \in \mathcal{N}_1^-(\mathbf{O})$ and therefore by (0.1.d), there exists $g_1 \in \mathrm{F}_{4(-20)}$ such that $(g_1X)^{\times 2} = g_1(X^{\times 2}) = P^-$ and $\mathrm{tr}(g_1X) = \mathrm{tr}(X) = 0$. Then $g_1X \in \mathcal{R}^-$. Thus, applying Lemma 5.5, there exists $g_2 \in (\mathrm{F}_{4(-20)})_{P^-}$ such that $g_2g_1X = Q^+(1)$. □

Remark 5.7. From (5.1.a), (0.1.a) and (0.1.b), we see that $\mathcal{H} = \mathcal{H}(\mathbf{O}) \coprod \mathcal{H}'(\mathbf{O}) = Orb_{\mathrm{F}_{4(-20)}}(E_1) \coprod Orb_{\mathrm{F}_{4(-20)}}(E_3)$ and that $\mathcal{H}$ consists of two $\mathrm{F}_{4(-20)}$-orbits. Then we can write $\mathcal{H} = \{X \in \mathcal{J}^1 |\ X \circ X = X,\ \mathrm{tr}(X) = 1\}$.

For $\xi = (\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3$ and $x = (x_1, x_2, x_3) \in \mathbf{O}^3$, put $h'(\xi; x) := \begin{pmatrix} r_1 & -\sqrt{-1}\overline{x_1} & -\sqrt{-1}\overline{x_2} \\ \sqrt{-1}x_1 & r_2 & x_3 \\ \sqrt{-1}x_2 & \overline{x_3} & r_3 \end{pmatrix}$. Denote the exceptional $\mathbb{R}$-Jordan algebra $\mathrm{Herm}'(3, \mathbf{O}) := \{h'(\xi; x)|\ \xi \in \mathbb{R}^3, x \in \mathbf{O}^3\}$ with the Jordan product $X \circ Y = 2^{-1}(XY + YX)$ $(X, Y \in \mathrm{Herm}'(3, \mathbf{O}))$. Put the linear Lie group $F_4' := \{g \in \mathrm{GL}_{\mathbb{R}}(\mathrm{Herm}'(3, \mathbf{O}))|\ g(X \circ Y) = gX \circ gY\}$ and the subset $\mathcal{H}' := \{A \in \mathrm{Herm}'(3, \mathbf{O})|\ A \circ A = A, \mathrm{tr}(A) = 1\}$ in $\mathrm{Herm}'(3, \mathbf{O})$, respectively. F.R. Harvey [6, page 296–297] mentions that F_4' is considered to be a simple Lie group of the type of $\mathbf{F}_{4(-20)}$ and $F_4'/\mathrm{Spin}(9) \simeq \mathcal{H}' = Orb_{F_4'}(E_1)$. Then $\mathcal{H}'$ consists of one F_4'-orbit. We notice that the linear Lie group $\mathrm{F}_4 := \{g \in \mathrm{GL}_{\mathbb{R}}(\mathcal{J})|\ g(X \circ Y) = gX \circ gY\}$ is a compact type of $\mathbf{F}_{4(-52)}$ with $\mathcal{J} = \{h(\xi, x)|\ \xi \in \mathbb{R}^3, x \in \mathbb{O}^3\}$ and that there exists an isomorphism $\Phi : \mathcal{J} \to \mathrm{Herm}'(3, \mathbf{O})$ as $\mathbb{R}$-Jordan algebra given as follows $\Phi(A) = \mathrm{diag}(-\sqrt{-1}, 1, 1)\ A\ \mathrm{diag}(-\sqrt{-1}, 1, 1)^{-1}$. Therefore, F_4' is a compact type of $\mathbf{F}_{4(-52)}$.

6. The orbit decomposition of $\mathcal{J}^1$ under $\mathrm{F}_{4(-20)}$.

In this section, we determine the orbit decomposition of $\mathcal{J}^1$ under the action of $\mathrm{F}_{4(-20)}$.

Lemma 6.1. *Let $\lambda_1 \in \mathbb{R}$. Then the following equations hold.*

$$((\lambda_1E - X)^{\times 2})^{\times 2} = \Phi_X(\lambda_1)(\lambda_1E - X), \tag{6.1.a}$$

$$\mathrm{tr}((\lambda_1E - X)^{\times 2}) = \left(\frac{d}{d\lambda}\Phi_X\right)(\lambda_1). \tag{6.1.b}$$

Proof. By (1.3.b)(iv),

$$((\lambda_1E - X)^{\times 2})^{\times 2} = \det(\lambda_1E - X)(\lambda_1E - X) = \Phi_X(\lambda_1)(\lambda_1E - X).$$

Using (1.3.a) and (1.3.b), the left side hand of (6.1.b) is

$$\mathrm{tr}(\lambda_1^2E^{\times 2} - 2\lambda_1(E \times X) + X^{\times 2}) = 3\lambda_1^2 - 2\mathrm{tr}(X)\lambda_1 + \mathrm{tr}(X^{\times 2})$$

and by (1.3.c), the right side hand of (6.1.b) is $3\lambda_1^2 - 2\mathrm{tr}(X)\lambda_1 + \mathrm{tr}(X^{\times 2})$. Thus (6.1.b) follows. □

Lemma 6.2. *Assume that $X \in \mathcal{J}^1$ has a characteristic root $\lambda_1 \in \mathbb{R}$ of multiplicity* 1. *Let $Z = \lambda_1 E - X$.*

(1) *The following assertions hold.*

(6.2.a)
$$\begin{cases} \text{(i)} & \det(Z) = 0, \quad \text{(ii) } \operatorname{tr}(Z^{\times 2}) \neq 0, \\ \text{(iii)} & (Z^{\times 2})^{\times 2} = 0, \\ \text{(iv)} & (Z^{\times 2}) \times Z = 2^{-1}(-\operatorname{tr}(Z)Z^{\times 2} - \operatorname{tr}(Z^{\times 2})Z + \operatorname{tr}(Z^{\times 2})\operatorname{tr}(Z)E). \end{cases}$$

(2) *The following equations hold.*

$$\text{(6.2.b)} \quad \begin{cases} \text{(i)} & E_{X,\lambda_1} = (\operatorname{tr}(Z^{\times 2}))^{-1} Z^{\times 2}, \\ \text{(ii)} & W_{X,\lambda_1} = (\operatorname{tr}(Z)/2)E - Z - (\operatorname{tr}(Z)/(2\operatorname{tr}(Z^{\times 2})))\, Z^{\times 2}, \\ \text{(iii)} & W_{X,\lambda_1}^{\times 2} = ((4\operatorname{tr}(Z^{\times 2}) - \operatorname{tr}(Z)^2)/(4\operatorname{tr}(Z^{\times 2})))\, Z^{\times 2}. \end{cases}$$

$$\text{(6.2.c)} \qquad Q(W_{X,\lambda_1}) = -4^{-1}(4\operatorname{tr}(Z^{\times 2}) - \operatorname{tr}(Z)^2).$$

Proof. (1) Since λ_1 is a characteristic root of X with multiplicity 1, we have $\det(Z) = \Phi_X(\lambda_1) = 0$ and $\left(\frac{d}{d\lambda}\Phi_X\right)(\lambda_1) \neq 0$. Thus (i) follows and (ii) follows from (6.1.b). By (6.1.a), $(Z^{\times 2})^{\times 2} = \Phi_X(\lambda_1)Z = 0$ so that (iii) follows. Moreover, (iv) follows from (i) and (1.3.b)(v).

(2) From (6.2.a)(ii), E_{X,λ_1} and W_{X,λ_1} are well-defined and (i) and (ii) follow from direct calculations. Thus, using Lemma 1.3 and (6.2.a), we calculate that

$$\begin{aligned} W_{X,\lambda_1}^{\times 2} &= \left((\operatorname{tr}(Z)/2)E - Z - \left(\operatorname{tr}(Z)/(2\operatorname{tr}(Z^{\times 2}))\right) Z^{\times 2}\right)^{\times 2} \\ &= \left((4\operatorname{tr}(Z^{\times 2}) - \operatorname{tr}(Z)^2)/(4\operatorname{tr}(Z^{\times 2}))\right) Z^{\times 2} \end{aligned}$$

and so $Q(W_{X,\lambda_1}) = -\operatorname{tr}(W_{X,\lambda_1}^{\times 2}) = -4^{-1}(4\operatorname{tr}(Z^{\times 2}) - \operatorname{tr}(Z)^2)$. □

Lemma 6.3. *Assume that $X \in \mathcal{J}^1$ has a characteristic root $\lambda_1 \in \mathbb{R}$ of multiplicity* 1. *Then*

$$\text{(6.3.a)} \qquad X = \lambda_1 E_{X,\lambda_1} + 2^{-1}(\operatorname{tr}(X) - \lambda_1)(E - E_{X,\lambda_1}) + W_{X,\lambda_1}$$

where

$$\text{(6.3.b)} \qquad E_{X,\lambda_1} \in \mathcal{H}(\mathbf{O}) \coprod \mathcal{H}'(\mathbf{O}),$$

$$\text{(6.3.c)} \qquad E - E_{X,\lambda_1} \in \mathcal{J}^1_{L^\times(2E_{X,\lambda_1}),1},$$

$$\text{(6.3.d)} \qquad W_{X,\lambda_1} \in \mathcal{J}^1_{L^\times(2E_{X,\lambda_1}),-1}.$$

Furthermore

$$\text{(6.3.e)} \qquad Q(W_{X,\lambda_1}) = -4^{-1}(3\lambda_1^2 - 2\lambda_1\operatorname{tr}(X) + 4\operatorname{tr}(X^{\times 2}) - \operatorname{tr}(X)^2).$$

Proof. Let $Z = \lambda_1 E - X$. First, (6.3.a) follows from (6.2.a)(ii) and (1.9). From (6.2.b)(i) and (6.2.a)(iii), we see that

$$E_{X,\lambda_1}^{\times 2} = \left(\operatorname{tr}(Z^{\times 2})\right)^{-2} (Z^{\times 2})^{\times 2} = 0,\ \operatorname{tr}(E_{X,\lambda_1}) = (\operatorname{tr}(Z^{\times 2}))^{-1}\operatorname{tr}(Z^{\times 2}) = 1$$

so that $E_{X,\lambda_1} \in \mathcal{H}$, and from (5.1.a), (6.3.b) follows. Second, because of $E_{X,\lambda_1} \in \mathcal{H}$ and (1.3.b)(ii), we see that

$$2E_{X,\lambda_1} \times (E - E_{X,\lambda_1}) = 2E_{X,\lambda_1} \times E = \mathrm{tr}(E_{X,\lambda_1})E - E_{X,\lambda_1} = E - E_{X,\lambda_1}$$

and $E - E_{X,\lambda_1} \in \mathcal{J}^1_{L^\times(2E_{X,\lambda_1}),1}$. Third, by (6.2.b)(i)(ii), $2E_{X,\lambda_1} \times W_{X,\lambda_1} = (2/\mathrm{tr}(Z^{\times 2}))Z^{\times 2} \times ((\mathrm{tr}(Z)/2)E - Z - (\mathrm{tr}(Z)/(2\mathrm{tr}(Z^{\times 2})))Z^{\times 2})$. Then by (1.3.b), (6.2.a)(iii) and (6.2.a)(iv), the right hand side is

$$-(\mathrm{tr}(Z)/2)E + Z + (\mathrm{tr}(Z)/(2\mathrm{tr}(Z^{\times 2})))Z^{\times 2} = -W_{X,\lambda_1}.$$

Thus $W_{X,\lambda_1} \in \mathcal{J}^1_{L^\times(2E_{X,\lambda_1}),-1}$. Last, using (6.2.c) and (6.1.b),

$$\begin{aligned} Q(W_{X,\lambda_1}) &= -4^{-1}(4(3\lambda_1^2 - 2\mathrm{tr}(X)\lambda_1 + \mathrm{tr}(X^{\times 2})) - (3\lambda_1 - \mathrm{tr}(X))^2) \\ &= -4^{-1}(3\lambda_1^2 - 2\lambda_1\mathrm{tr}(X) + 4\mathrm{tr}(X^{\times 2}) - \mathrm{tr}(X)^2). \end{aligned}$$

□

Lemma 6.4. *Assume that $X \in \mathcal{J}^1$ has a characteristic root $\lambda_1 \in \mathbb{R}$ of multiplicity* 1. *Then $E_{X,\lambda_1} \in \mathcal{H}(\mathbf{O})$ or $E_{X,\lambda_1} \in \mathcal{H}'(\mathbf{O})$ and the following* (i) *or* (ii) *hold.*

(i) *When $E_{X,\lambda_1} \in \mathcal{H}(\mathbf{O})$, then there exists $g \in \mathrm{F}_{4(-20)}$ such that*

$$gX = \lambda_1 E_1 + 2^{-1}(\mathrm{tr}(X) - \lambda_1)(E - E_1) + gW_{X,\lambda_1}$$

where $gW_{X,\lambda_1} \in \mathcal{J}^1_{L^\times(2E_1),-1}$ and the quadratic space $(\mathcal{J}^1_{L^\times(2E_{X,\lambda_1}),-1}, Q)$ is isomorphic to $(\mathbb{R}^{0,9}, \mathrm{q}_{0,9})$. Especially, $Q(W_{X,\lambda_1}) \geq 0$ and

$$Q(W_{X,\lambda_1}) = 0 \quad \textit{iff} \quad W_{X,\lambda_1} = 0.$$

(ii) *When $E_{X,\lambda_1} \in \mathcal{H}'(\mathbf{O})$, then there exists $g \in \mathrm{F}_{4(-20)}$ such that*

$$gX = \lambda_1 E_3 + 2^{-1}(\mathrm{tr}(X) - \lambda_1)(E - E_3) + gW_{X,\lambda_1}$$

where $gW_{X,\lambda_1} \in \mathcal{J}^1_{L^\times(2E_3),-1}$ and the quadratic space $(\mathcal{J}^1_{L^\times(2E_{X,\lambda_1}),-1}, Q)$ isomorphic to $(\mathbb{R}^{8,1}, \mathrm{q}_{8,1})$.

Proof. By Lemma 6.3,

$$X = \lambda_1 E_{X,\lambda_1} + 2^{-1}(\mathrm{tr}(X) - \lambda_1)(E - E_{X,\lambda_1}) + W_{X,\lambda_1}$$

where $E_{X,\lambda_1} \in \mathcal{H}(\mathbf{O}) \coprod \mathcal{H}'(\mathbf{O})$ and $W_{X,\lambda_1} \in \mathcal{J}^1_{L^\times(2E_{X,\lambda_1}),-1}$.

Case $E_{X,\lambda_1} \in \mathcal{H}(\mathbf{O})$. By (0.1.a), there exists $g \in \mathrm{F}_{4(-20)}$ such that $gE_{X,\lambda_1} = E_1$ and it is clear that

$$gX = \lambda_1 E_1 + 2^{-1}(\mathrm{tr}(X) - \lambda_1)(E - E_1) + gW_{X,\lambda_1}.$$

By (4.1.b), $gW_{X,\lambda_1} \in g\mathcal{J}^1_{L^\times(2E_{X,\lambda_1}),-1} = \mathcal{J}^1_{L^\times(2E_1),-1}$. From Lemma 4.1, we see that g gives the quadratic isomorphism from $(\mathcal{J}^1_{L^\times(2E_{X,\lambda_1}),-1}, Q)$ onto $(\mathcal{J}^1_{L^\times(2E_1),-1}, Q)$ and that the quadratic space $(\mathcal{J}^1_{L^\times(2E_{X,\lambda_1}),-1}, Q)$ is isomorphic to $(\mathbb{R}^{0,9}, \mathrm{q}_{0,9})$.

Case $E_{X,\lambda_1} \in \mathcal{H}'(\mathbf{O})$. By (0.1.b), there exists $g \in \mathrm{F}_{4(-20)}$ such that $gE_{X,\lambda_1} = E_3$ and as similar to (i), we see

$$gX = \lambda_1 E_3 + 2^{-1}(\mathrm{tr}(X) - \lambda_1)(E - E_3) + gW_{X,\lambda_1}$$

with $gW_{X,\lambda_1} \in \mathcal{J}^1_{L^\times(2E_3),-1}$. From Lemma 4.1, we see that g gives the quadratic isomorphism from $(\mathcal{J}^1_{L^\times(2E_{X,\lambda_1}),-1}, Q)$ onto $(\mathcal{J}^1_{L^\times(2E_3),-1}, Q)$ and that $(\mathcal{J}^1_{L^\times(2E_{X,\lambda_1}),-1}, Q)$ is isomorphic to $(\mathbb{R}^{8,1}, \mathrm{q}_{8,1})$. $\square$

Lemma 6.5. *Assume that* $X \in \mathcal{J}^1$ *has a characteristic root* $\lambda_1 \in \mathbb{R}$ *of multiplicity* 1 *and* $Q(W_{X,\lambda_1}) > 0$. *Then* X *is diagonalizable under the action of* $\mathrm{F}_{4(-20)}$ *on* $\mathcal{J}^1$.

Proof. By Lemma 6.4, there exists $g_1 \in \mathrm{F}_{4(-20)}$ such that

$$g_1 X = \lambda_1 E_i + 2^{-1}(\mathrm{tr}(X) - \lambda_1)(E - E_i) + g_1 W_{X,\lambda_1}$$

where $g_1 W_{X,\lambda_1} \in \mathcal{J}^1_{L^\times(2E_i),-1}$ and $i \in \{1, 3\}$. From (5.1.a), we see $Q(g_1 W_{X,\lambda_1}) = Q(W_{X,\lambda_1}) > 0$ and therefore $g_1 W_{X,\lambda_1}$ can be expressed by $g_1 W_{X,\lambda_1} = \sqrt{Q(W_{X,\lambda_1})}Y$ where $Y \in \mathcal{J}^1_{L^\times(2E_i),-1}$ with $Q(Y) = 1$. If $i = 1$ then $Y \in S^8$, and if $i = 3$ then $Y \in \mathcal{S}^{8,1} = \mathcal{S}^{8,1}_+ \coprod \mathcal{S}^{8,1}_-$. Then we have the following three cases: (i) $i = 1$, (ii) $i = 3$ and $Y \in \mathcal{S}^{8,1}_+$, (iii) $i = 3$ and $Y \in \mathcal{S}^{8,1}_-$.

Case (i): $i = 1$. By (4.6.a), there exists $g_2 \in (\mathrm{F}_{4(-20)})_{E_1}$ such that $g_2 Y = E_2 - E_3$ and it is clear that

$$g_2 g_1 X = \lambda_1 E_1 + \sum{}_{j=2}^{3} \left(2^{-1}(\mathrm{tr}(X) - \lambda_1) + (-1)^j \sqrt{Q(W_{X,\lambda_1})}\right) E_j.$$

Case (ii): $i = 3$ and $Y \in \mathcal{S}^{8,1}_+$. By (4.6.c)(i), there exists $g_2' \in (\mathrm{F}_{4(-20)})_{E_3}$ such that $g_2' Y = E_1 - E_2$ and it is clear that

$$g_2' g_1 X = \lambda_1 E_3 + \sum{}_{j=1}^{2} \left(2^{-1}(\mathrm{tr}(X) - \lambda_1) + (-1)^{j+1} \sqrt{Q(W_{X,\lambda_1})}\right) E_j.$$

Case (iii): $i = 3$ and $Y \in \mathcal{S}^{8,1}_-$. By (4.6.c)(ii), there exists $g_2'' \in (\mathrm{F}_{4(-20)})_{E_3}$ such that $g_2'' Y = -E_1 + E_2$ and it is clear that

$$g_2'' g_1 X = \lambda_1 E_3 + \sum{}_{j=1}^{2} \left(2^{-1}(\mathrm{tr}(X) - \lambda_1) + (-1)^j \sqrt{Q(W_{X,\lambda_1})}\right) E_j.$$

Thus these cases imply that X can be transformed to a diagonal matrix under the action of $\mathrm{F}_{4(-20)}$. $\square$

Lemma 6.6. *Assume that* $X \in \mathcal{J}^1$ *has a characteristic polynomial* $\Phi_X(\lambda) = (\lambda - \lambda_1)(\lambda - \lambda_2)(\lambda - \lambda_3)$ *where* $\lambda_i \in \mathbb{C}$. *Then*

(6.6)
$$\begin{cases} \text{(i)} \ \ \mathrm{tr}(X) = \lambda_1 + \lambda_2 + \lambda_3, \ \ \text{(ii)} \ \ \mathrm{tr}(X^{\times 2}) = \lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1, \\ \text{(iii)} \ \ \det(X) = \lambda_1\lambda_2\lambda_3. \end{cases}$$

Proof. By (1.3.c), $\lambda^3 - \mathrm{tr}(X)\lambda^2 + \mathrm{tr}(X^{\times 2})\lambda - \det(X) = \Phi_X(\lambda) = \lambda^3 - (\lambda_1 + \lambda_2 + \lambda_3)\lambda^2 + (\lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1)\lambda - \lambda_1\lambda_2\lambda_3$. Thus the result follows. $\square$

Lemma 6.7. *Assume that $X \in \mathcal{J}^1$ has a characteristic root $\lambda_1 \in \mathbb{R}$ of multiplicity 1 and $\Phi_X(\lambda) = (\lambda - \lambda_1)(\lambda - \lambda_2)(\lambda - \lambda_3)$ where $\lambda_2, \lambda_3 \in \mathbb{C}$. Then*

$$Q(W_{X,\lambda_1}) = 4^{-1}(\lambda_2 - \lambda_3)^2. \tag{6.7}$$

Proof. By (6.3.c) and (6.6), $Q(W_{X,\lambda_1}) = -4^{-1}(3\lambda_1^2 - 2(\lambda_1 + \lambda_2 + \lambda_3)\lambda_1 + 4(\lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1) - (\lambda_1 + \lambda_2 + \lambda_3)^2) = 4^{-1}(\lambda_2 - \lambda_3)^2$. □

Lemma 6.8. *Let real numbers r_1, r_2, r_3 be different from each other and $Y = \mathrm{diag}(r_1, r_2, r_3) \in \mathcal{J}^1$.*

(1) *All of characteristic roots of Y are r_1, r_2, r_3.*

(2) *$E_i = E_{Y,r_i}$ for all $i \in \{1, 2, 3\}$.*

(3) *$V_Y = \{aE_1 + bE_2 + cE_3 |\ a, b, c \in \mathbb{R}\}$.*

(4) *$\mathcal{H}(\mathbf{O}) \cap V_Y = \{E_{Y,r_1}\}$ and $\mathcal{H}'(\mathbf{O}) \cap V_Y = \{E_{Y,r_2},\ E_{Y,r_3}\}$ with $E_{Y,r_2} \neq E_{Y,r_3}$.*

(5) *For any $g \in \mathrm{F}_{4(-20)}$, $\mathcal{H}(\mathbf{O}) \cap V_{gY} = \{E_{gY,r_1}\}$ and $\mathcal{H}'(\mathbf{O}) \cap V_{gY} = \{E_{gY,r_2},\ E_{gY,r_3}\}$ with $E_{gY,r_2} \neq E_{gY,r_3}$.*

Proof. (1) By (1.6.c), $\Phi_Y(\lambda) = (\lambda - r_1)(\lambda - r_2)(\lambda - r_3)$. Hence (1) follows.

(2) Because of $(r_iE - Y)^{\times 2} = (r_i - r_{i+1})(r_i - r_{i+2})E_i$, we see $E_{Y,r_i} = E_i$.

(3) Put the linear space $V = \{aE_1 + bE_2 + cE_3 |\ a, b, c \in \mathbb{R}\}$. Because of (2) and $E_{Y,r_i} \in V_Y$, we see $E_1, E_2, E_3 \in V_Y$ so that $V \subset V_Y$. Thus it follows from $3 = \dim V \leq \dim V_Y \leq 3$.

(4) Let $Z \in V_Y \cap \mathcal{H}$. By (3), Z can be expressed by $Z = aE_1 + bE_2 + cE_3$ for some $a, b, c \in \mathbb{R}$. Because of $aE_1 + bE_2 + cE_3 \in \mathcal{H}$, we see that $0 = (aE_1 + bE_2 + cE_3)^{\times 2} = bcE_1 + caE_2 + abE_3$ and $1 = \mathrm{tr}(aE_1 + bE_2 + cE_3) = a + b + c$. Then $bc = ca = ab = 0$ and $a + b + c = 1$. Solving these equations, $(a, b, c) = (1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$ and it is clear that $\mathcal{H} \cap V_Y = \{E_1, E_2, E_3\}$. Thus $\mathcal{H}(\mathbf{O}) \cap V_Y = \{E_1\} = \{E_{Y,r_1}\}$ because $(E_i|E_1) \geq 1$ iff $i = 1$, and $\mathcal{H}'(\mathbf{O}) \cap V_Y = \{E_2, E_3\} = \{E_{Y,r_2}, E_{Y,r_3}\}$ because $(E_i|E_1) \leq 0$ iff $i = 2, 3$. Moreover $E_{Y,r_2} = E_2 \neq E_3 = E_{Y,r_3}$. Hence (4) follows.

(5) From (0.1.a), (4) and (1.10)(iii), it follows that $\mathcal{H}(\mathbf{O}) \cap V_{gY} = (g\mathcal{H}(\mathbf{O})) \cap g(V_X) = g(\mathcal{H}(\mathbf{O}) \cap V_Y) = \{E_{gY,\lambda_i}\}$ and $\mathcal{H}'(\mathbf{O}) \cap V_{gY} = (g\mathcal{H}'(\mathbf{O})) \cap g(V_X) = g(\mathcal{H}'(\mathbf{O}) \cap V_Y) = \{E_{gY,\lambda_{i+1}},\ E_{gY,\lambda_{i+2}}\}$ with $E_{gY,r_2} \neq E_{gY,r_3}$. □

Lemma 6.9. *Let $i \in \{1, 2, 3\}$ and indexes $i, i+1, i+2$ be counted modulo 3. Let real numbers λ_1, λ_2, λ_3 be different from each other and $Y_i = \mathrm{diag}(\lambda_i, \lambda_{i+1}, \lambda_{i+2}) \in \mathcal{J}^1$. Then orbits $Orb_{\mathrm{F}_{4(-20)}}(Y_1)$, $Orb_{\mathrm{F}_{4(-20)}}(Y_2)$, $Orb_{\mathrm{F}_{4(-20)}}(Y_3)$ are different orbits from each other.*

Proof. It is enough to show $Orb_{\mathrm{F}_{4(-20)}}(Y_i) \neq Orb_{\mathrm{F}_{4(-20)}}(Y_{i+1})$. Suppose that there exists $g \in \mathrm{F}_{4(-20)}$ such that $gY_i = Y_{i+1}$. By Lemma 6.8(4),

$$\begin{aligned} \mathcal{H}(\mathbf{O}) \cap V_{Y_i} &= \{E_{Y_i,\lambda_i}\}, & \mathcal{H}'(\mathbf{O}) \cap V_{Y_i} &= \{E_{Y_i,\lambda_{i+1}},\ E_{Y_i,\lambda_{i+2}}\}, \\ \mathcal{H}(\mathbf{O}) \cap V_{Y_{i+1}} &= \{E_{Y_{i+1},\lambda_{i+1}}\}, & \mathcal{H}'(\mathbf{O}) \cap V_{Y_{i+1}} &= \{E_{Y_{i+1},\lambda_{i+2}},\ E_{Y_{i+1},\lambda_i}\}. \end{aligned}$$

In particular, $E_{Y_i,\lambda_{i+1}} \in \mathcal{H}'(\mathbf{O})$ and $E_{Y_{i+1},\lambda_{i+1}} \in \mathcal{H}(\mathbf{O})$. From (0.1.a) and (1.10)(iii), we see $E_{Y_{i+1},\lambda_{i+1}} = E_{gY_i,\lambda_{i+1}} = gE_{Y_i,\lambda_{i+1}} \in g\mathcal{H}'(\mathbf{O}) = \mathcal{H}'(\mathbf{O})$. Then from(5.1.a), it follows that $E_{Y_{i+1},\lambda_{i+1}} \in \mathcal{H}(\mathbf{O}) \cap \mathcal{H}'(\mathbf{O}) = \emptyset$. It is a contradiction as required. $\square$

Proposition 6.10. *Let $i \in \{1, 2, 3\}$ and indexes i, $i + 1$, $i + 2$ be counted modulo 3. Assume that $X \in \mathcal{J}^1$ admits characteristic roots $\lambda_1 > \lambda_2 > \lambda_3$.*

(1) *There exist the unique $i \in \{1, 2, 3\}$ such that*

$$X \in Orb_{\mathrm{F}_{4(-20)}}(\mathrm{diag}(\lambda_i, \lambda_{i+1}, \lambda_{i+2})).$$

(2) *The following assertions are equivalent.*

(i) $X \in Orb_{\mathrm{F}_{4(-20)}}(\mathrm{diag}(\lambda_i, \lambda_{i+1}, \lambda_{i+2}))$.

(ii) $\begin{cases} \mathcal{H}(\mathbf{O}) \cap V_X = \{E_{X,\lambda_i}\}, \\ \mathcal{H}'(\mathbf{O}) \cap V_X = \{E_{X,\lambda_{i+1}},\ E_{X,\lambda_{i+2}}\} \text{ with } E_{X,\lambda_{i+1}} \neq E_{X,\lambda_{i+2}}. \end{cases}$

Proof. (1) Fix the characteristic root $\lambda_1 \in \mathbb{R}$. Because of $0 \neq \lambda_2 - \lambda_3 \in \mathbb{R}$ and (6.7), we see $Q(W_{X,\lambda_1}) = 4^{-1}(\lambda_2 - \lambda_3)^2 > 0$. By Lemma 6.5, we see that X is diagonalizable under the action of $\mathrm{F}_{4(-20)}$. Then from Lemma 6.8(1), there exists $i \in \{1, 2, 3\}$ and $g_0 \in \mathrm{F}_{4(-20)}$ such that $g_0X = \mathrm{diag}(\lambda_i, \lambda_{i+1}, \lambda_{i+2})$ or $g_0X = \mathrm{diag}(\lambda_i, \lambda_{i+2}, \lambda_{i+1})$. When $g_0X = \mathrm{diag}(\lambda_i, \lambda_{i+2}, \lambda_{i+1})$, from (3.10.a), we see $\exp(2^{-1}\pi\tilde{A}^1_1(1))g_0X = \mathrm{diag}(\lambda_i, \lambda_{i+1}, \lambda_{i+2})$. Thus $X \in Orb_{F_{4(-20}}(\mathrm{diag}(\lambda_i, \lambda_{i+1}, \lambda_{i+2}))$ and by Lemma 6.9, such i is unique. Hence (1) follows.

(2) We show (i) $\Rightarrow$ (ii). Then it implies that (i) and (ii) are equivalent (a proof by transposition). Suppose $gX = \mathrm{diag}(\lambda_i, \lambda_{i+1}, \lambda_{i+2})$ for some $g \in \mathrm{F}_{4(-20)}$. Since gX is diagonal matrix, from Lemma 6.8(4), we see $\mathcal{H}(\mathbf{O}) \cap V_{gX} = \{E_{gX,\lambda_i}\}$ and $\mathcal{H}'(\mathbf{O}) \cap V_{gX} = \{E_{gX,\lambda_{i+1}},\ E_{gX,\lambda_{i+2}}\}$ with $E_{gX,\lambda_{i+1}} \neq E_{gX,\lambda_{i+2}}$. Then from Lemma 6.8(5), it follows that $\mathcal{H}(\mathbf{O}) \cap V_X = \mathcal{H}(\mathbf{O}) \cap V_{g^{-1}gX} = \{E_{g^{-1}gX,\lambda_i}\} = \{E_{X,\lambda_i}\}$ and $\mathcal{H}'(\mathbf{O}) \cap V_X = \mathcal{H}'(\mathbf{O}) \cap V_{g^{-1}gX} = \{E_{X,\lambda_{i+1}},\ E_{X,\lambda_{i+2}}\}$ with $E_{X,\lambda_{i+1}} \neq E_{X,\lambda_{i+2}}$. $\square$

Proposition 6.11. *Assume that $X \in \mathcal{J}^1$ admits characteristic roots $\lambda_1 \in \mathbb{R}$, $p \pm \sqrt{-1}q$ with $p \in \mathbb{R}$ and $q > 0$. Then*

$$X \in Orb_{\mathrm{F}_{4(-20)}}(\mathrm{diag}(p, p, \lambda_1) + F^1_3(q)).$$

Proof. Since λ_1 is a characteristic root in $\mathbb{R}$ of multiplicity 1, from (6.7), we see $Q(W_{X,\lambda_1}) = 4^{-1}((p + \sqrt{-1}q) - (p - \sqrt{-1}q))^2 = -q^2 < 0$.

From Lemma 6.4, we see that E_{X,λ_1} must be in $\mathcal{H}'(\mathbf{O})$ and there exists $g_1 \in \mathrm{F}_{4(-20)}$ such that

$$g_1 X = \lambda_1 E_3 + 2^{-1}(\mathrm{tr}(X) - \lambda_1)(E - E_3) + g_1 W_{X,\lambda_1}$$

where $g_1 W_{X,\lambda_1} \in \mathcal{J}^1_{L^\times(2E_3),-1}$ and $Q(g_1 W_{X,\lambda_1}) = Q(W_{X,\lambda_1}) = -q^2 < 0$. By (6.6)(i), $\mathrm{tr}(X) = \lambda_1 + (p + \sqrt{-1}q) + (p - \sqrt{-1}q) = \lambda_1 + 2p$ and therefore $g_1 X$ can be expressed by

$$g_1 X = \lambda_1 E_3 + p(E - E_3) + qY \quad \text{for some } Y \in \mathcal{S}^{8,1}(-1).$$

Now by (4.6.b), there exists $g_2 \in (\mathrm{F}_{4(-20)})_{E_3}$ such that $g_2 Y = F^1_3(1)$ and it is clear that

$$g_2 g_1 X = \lambda_1 E_3 + p(E - E_3) + F^1_3(q) = \mathrm{diag}(p, p, \lambda_1) + F^1_3(q).$$

□

Lemma 6.12.

$$\mathcal{N}^+_1(\mathbf{O}) \cap \mathcal{J}^1_{L^\times(2E_3),-1} = \mathcal{N}^{8,1}_+, \quad \mathcal{N}^-_1(\mathbf{O}) \cap \mathcal{J}^1_{L^\times(2E_3),-1} = \mathcal{N}^{8,1}_-.$$

Proof. Take $X \in \mathcal{N}^+_1(\mathbf{O}) \cap \mathcal{J}^1_{L^\times(2E_3),-1}$. Because of $X \in \mathcal{N}^+_1(\mathbf{O})$, $Q(X^{\times 2}) = 0$ and $(X|E_1) > 0$. Thus, $X \in \mathcal{N}^{8,1}_+$ and so $\mathcal{N}^+_1(\mathbf{O}) \cap \mathcal{J}^1_{L^\times(2E_3),-1} \subset \mathcal{N}^{8,1}_+$. Conversely, take $X \in \mathcal{N}^{8,1}_+$. Then X can be expressed by $X = \xi(E_1 - E_2) + F^1_3(x)$ where $\xi = (X|E_1) > 0$, $x \in \mathbf{O}$ and $0 = Q(X) = \xi^2 - (x|x)$. Now $X^{\times 2} = -(\xi^2 - (x|x))E_3 = 0$, $\mathrm{tr}(X) = 0$ and $(X|E_1) > 0$. Thus $X \in \mathcal{N}^+_1(\mathbf{O}) \cap \mathcal{J}^1_{L^\times(2E_3),-1}$ and so $\mathcal{N}^{8,1}_+ \subset \mathcal{N}^+_1(\mathbf{O}) \cap \mathcal{J}^1_{L^\times(2E_3),-1}$. Hence $\mathcal{N}^+_1(\mathbf{O}) \cap \mathcal{J}^1_{L^\times(2E_3),-1} = \mathcal{N}^{8,1}_+$. Similarly, we obtain $\mathcal{N}^-_1(\mathbf{O}) \cap \mathcal{J}^1_{L^\times(2E_3),-1} = \mathcal{N}^{8,1}_-$. □

Proposition 6.13. *Assume that $X \in \mathcal{J}^1$ admits characteristic roots λ_1 of multiplicity 1 and λ_2 of multiplicity 2. Then we have*

(i) $E_{X,\lambda_1} \in \mathcal{H}(\mathbf{O}) \coprod \mathcal{H}'(\mathbf{O})$, (ii) $W_{X,\lambda_1} \in \{0\} \coprod \mathcal{N}^+_1(\mathbf{O}) \coprod \mathcal{N}^-_1(\mathbf{O})$,
(iii) $E_{X,\lambda_1} \in \mathcal{H}(\mathbf{O}) \Rightarrow W_{X,\lambda_1} = 0$ (iv) $W_{X,\lambda_1} \neq 0 \Rightarrow E_{X,\lambda_1} \in \mathcal{H}'(\mathbf{O})$

and the following assertions hold:

(1) $E_{X,\lambda_1} \in \mathcal{H}(\mathbf{O}) \Rightarrow X \in Orb_{\mathrm{F}_{4(-20)}}(\mathrm{diag}(\lambda_1, \lambda_2, \lambda_2))$,
(2) $E_{X,\lambda_1} \in \mathcal{H}'(\mathbf{O}),\ W_{X,\lambda_1} = 0 \Rightarrow X \in Orb_{\mathrm{F}_{4(-20)}}(\mathrm{diag}(\lambda_2, \lambda_2, \lambda_1))$,
(3) $W_{X,\lambda_1} \in \mathcal{N}^+_1(\mathbf{O}) \Rightarrow X \in Orb_{\mathrm{F}_{4(-20)}}(\mathrm{diag}(\lambda_2, \lambda_2, \lambda_1) + P^+)$,
(4) $W_{X,\lambda_1} \in \mathcal{N}^-_1(\mathbf{O}) \Rightarrow X \in Orb_{\mathrm{F}_{4(-20)}}(\mathrm{diag}(\lambda_2, \lambda_2, \lambda_1) + P^-)$.

Proof. (i) follows from (6.3.b). Since λ_1 is a characteristic root in $\mathbb{R}$ of multiplicity 1, form (6.7), we see $Q(W_{X,\lambda_1}) = 4^{-1}(\lambda_2 - \lambda_2)^2 = 0$. Put $Z = \lambda_1 E - X$. From (6.2.b)(iii) and (6.2.c), we see

$$W^{\times 2}_{X,\lambda_1} = -(Q(W_{X,\lambda_1})/\mathrm{tr}(Z^{\times 2}))Z^{\times 2} = 0.$$

Now, because of $\mathrm{tr}(E_{X,\lambda_1}) = 1$ and $\mathrm{tr}(E) = 3$, we see

$$\mathrm{tr}(W_{X,\lambda_1}) = \mathrm{tr}\left(X - \left(\lambda_1 E_{X,\lambda_1} + 2^{-1}(\mathrm{tr}(X) - \lambda_1)(E - E_{X,\lambda_1})\right)\right) = 0.$$

Thus $W_{X,\lambda_1} \in \{0\} \coprod \mathcal{N}_1(\mathbf{O})$. Hence (ii) follows from (5.1.b), and (iii) follows from $Q(W_{X,\lambda_1}) = 0$ and Lemma 6.4(i). Moreover, it follows from (i) that (iv) is the contrapositive proposition of (iii).

(1) By (iii), $W_{X,\lambda_1} = 0$. Hence it follows from Lemma 6.4(i).

(2) It follows from Lemma 6.4(ii).

(3) From (iv), we see $E_{X,\lambda_1} \in \mathcal{H}'(\mathbf{O})$ and from Lemma 6.4, there exists $g_1 \in \mathrm{F}_{4(-20)}$ such that

$$g_1 X = \lambda_1 E_3 + 2^{-1}(\mathrm{tr}(X) - \lambda_1)(E - E_3) + g_1 W_{X,\lambda_1}$$

where $g_1 W_{X,\lambda_1} \in \mathcal{J}^1_{L^\times(2E_3),-1}$. By (0.1.d), $g_1 W_{X,\lambda_1} \in g_1 \mathcal{N}_1^+(\mathbf{O}) = \mathcal{N}_1^+(\mathbf{O})$ and by Lemma 6.12, $g_1 W_{X,\lambda_1} \in \mathcal{N}_1^+(\mathbf{O}) \cap \mathcal{J}^1_{L^\times(2E_3),-1} = \mathcal{N}_+^{8,1}$. From (4.6.d)(i), there exists $g_2 \in (\mathrm{F}_{4(-20)})_{E_3}$ such that $g_2 g_1 W_{X,\lambda_1} = P^+$. Now by (6.6)(i), $\mathrm{tr}(X) = \lambda_1 + 2\lambda_2$. Thus

$$g_2 g_1 X = \lambda_1 E_3 + 2^{-1}(\lambda_1 + 2\lambda_2 - \lambda_1)(E - E_3) + P^+ = \mathrm{diag}(\lambda_2, \lambda_2, \lambda_1) + P^+.$$

Hence (3) follows and similarly, (4) follows. □

Proposition 6.14. *Assume that $X \in \mathcal{J}^1$ admits a characteristic root of multiplicity 3. Then*

$$p(X) \in \{0\} \coprod \mathcal{N}_1^+(\mathbf{O}) \coprod \mathcal{N}_1^-(\mathbf{O}) \coprod \mathcal{N}_2(\mathbf{O})$$

and the following assertions hold:

(1) $p(X) = 0 \Rightarrow X \in Orb_{\mathrm{F}_{4(-20)}}(3^{-1}\mathrm{tr}(X)E)$,

(2) $p(X) \in \mathcal{N}_1^+(\mathbf{O}) \Rightarrow X \in Orb_{\mathrm{F}_{4(-20)}}(3^{-1}\mathrm{tr}(X)E + P^+)$,

(3) $p(X) \in \mathcal{N}_1^-(\mathbf{O}) \Rightarrow X \in Orb_{\mathrm{F}_{4(-20)}}(3^{-1}\mathrm{tr}(X)E + P^-)$,

(4) $p(X) \in \mathcal{N}_2(\mathbf{O}) \Rightarrow X \in Orb_{\mathrm{F}_{4(-20)}}(3^{-1}\mathrm{tr}(X)E + Q^+(1))$.

Proof. Put $Z = p(X)$. Because of $\Phi_X(\lambda) = (\lambda - 3^{-1}\mathrm{tr}(X))^3$, we see $\Phi_Z(\mu) = \det((\mu + 3^{-1}\mathrm{tr}(X))E - X) = \Phi_X(\mu + 3^{-1}\mathrm{tr}(X)) = \mu^3$. From (1.3.c), we see $\mathrm{tr}(Z) = \mathrm{tr}(Z^{\times 2}) = \det(Z) = 0$. Then by (5.3), $Z \in \mathcal{N} = \{0\} \coprod \mathcal{N}_1^+(\mathbf{O}) \coprod \mathcal{N}_1^-(\mathbf{O}) \coprod \mathcal{N}_2(\mathbf{O})$. Now $X = 3^{-1}\mathrm{tr}(X)E + p(X)$. Since E is invariant under the action of $\mathrm{F}_{4(-20)}$, (1),(2),(3) and (4) follows from (0.1.c), (0.1.d) and (0.1.e). □

Proof of Main Theorem 1. Fix $X \in \mathcal{J}^1$. $\Phi_X(\lambda)$ is a $\mathbb{R}$-coefficient polynomial of λ with degree 3, so that we obtain the following cases (1)-(4) by means of the set of all characteristic roots with multiplicities.

(1) $X \in \mathcal{J}^1$ admits characteristic roots $\lambda_1 > \lambda_2 > \lambda_3$.

(2) $X \in \mathcal{J}^1$ admits characteristic roots $\lambda_1 \in \mathbb{R}$ and $p \pm \sqrt{-1}q$ with $p \in \mathbb{R}$ and $q > 0$.

(3) $X \in \mathcal{J}^1$ admits characteristic roots λ_1 of multiplicity 1 and λ_2 of multiplicity 2.

(4) $X \in \mathcal{J}^1$ admits a characteristic root of multiplicity 3.

Because the set of all characteristic roots with multiplicities are invariant under the action of $\mathrm{F}_{4(-20)}$, the difference of set of all characteristic roots with multiplicities induces the difference of $\mathrm{F}_{4(-20)}$-orbits in $\mathcal{J}^1$. Therefore cases (1)-(4) are different cases of orbits from each other.

In case (1), by Proposition 6.10, we obtain the propositions (1)(i)-(ii) and the canonical forms. By Lemma 6.9, the three cases are different orbits from each other.

In case (2), by Proposition 6.11, we obtain the canonical form of X.

In case (3), from Proposition 6.13, we obtain the propositions (3)(i)-(iv) and the canonical forms. Put $\mathcal{O} = \mathcal{N}_1^+(\mathbf{O})$, $\mathcal{N}_1^+(\mathbf{O})$ or $\{0\}$. From (1.10) and Proposition 0.1, it follows that if $W_{X,\lambda_1} \in \mathcal{O}$ then

$$W_{gX,\lambda_1} = gW_{X,\lambda_1} \in g\mathcal{O} = \mathcal{O} \quad \text{for all } g \in \mathrm{F}_{4(-20)}.$$

Thus the condition $W_{X,\lambda_1} \in \mathcal{O}$ is invariant under the action of $\mathrm{F}_{4(-20)}$. Similarly, from (1.10) and Proposition 0.1, the condition $E_{X,\lambda_1} \in \mathcal{H}(\mathbf{O})$ or $E_{X,\lambda_1} \in \mathcal{H}'(\mathbf{O})$ is invariant under the action of $\mathrm{F}_{4(-20)}$. It implies that these four cases are different orbits from each other.

In case (4), by Proposition 6.14, we obtain the proposition and the canonical forms. Put $\mathcal{O}' = \mathcal{N}_1^+(\mathbf{O})$, $\mathcal{N}_1^+(\mathbf{O})$, $\mathcal{N}_2(\mathbf{O})$ or $\{0\}$. From (1.10) and Proposition 0.1, it follows that if $p(X) \in \mathcal{O}'$ then

$$p(gX) = gp(X) \in g\mathcal{O}' = \mathcal{O}' \quad \text{for all } g \in \mathrm{F}_{4(-20)}.$$

Thus the condition $p(X) \in \mathcal{O}'$ is invariant under the action of $\mathrm{F}_{4(-20)}$. It implies that these four cases are different orbits from each other.

Hence we obtain a concrete orbit decomposition of $\mathcal{J}^1$ under the action of $\mathrm{F}_{4(-20)}$. $\square$

7. The construction of nilpotent subgroup.

In this section, we explain the construction of nilpotent groups N^+ and N^-. The differential $d\tilde{\sigma}_i \in \mathrm{Aut}_{\mathbb{R}}(\mathfrak{f}_{4(-20)})$ of the involutive automorphism $\tilde{\sigma}_i$ is written by same letter $\tilde{\sigma}_i : \tilde{\sigma}_i\phi = \sigma_i\phi\sigma_i$ for $\phi \in \mathfrak{f}_{4(-20)}$.

Lemma 7.1. *Let $i \in \{1,2,3\}$ and indexes $i, i+1, i+2$ be counted modulo* 3. *Let $D \in \mathfrak{d}_4$ and $a, b \in \mathbf{O}$.*

(1) *The following equations hold.*

$$\text{(7.1.a)} \qquad \begin{cases} \text{(i)} \quad \tilde{\sigma}_i D = D, & \text{(ii)} \quad \tilde{\sigma}_i \tilde{A}_i^1(a) = \tilde{A}_i^1(a), \\ \text{(iii)} \quad \tilde{\sigma}_i \tilde{A}_j^1(a) = -\tilde{A}_j^1(a) & \text{for } j = i+1, i+2. \end{cases}$$

(2) *The following equations hold.*

$$\text{(7.1.b)} \qquad \begin{cases} \text{(i)} \quad [D, \tilde{A}_i^1(a)] = \tilde{A}_i^1(D_i a), \quad \text{(ii)} \quad [\tilde{A}_i^1(a), \tilde{A}_i^1(b)] \in \mathfrak{d}_4, \\ \text{(iii)} \quad [\tilde{A}_i^1(a), \tilde{A}_{i+1}^1(b)] = \epsilon(i+2)\tilde{A}_{i+2}^1(\overline{ab}) \end{cases}$$

where $D = d\varphi_0(D_1, D_2, D_3) \in \mathfrak{d}_4$ (see Lemma 3.5(3)).

Proof. Fix $X \in \{E_i, F_i^1(x) \mid x \in \mathbf{O},\ i = 1, 2, 3\}$.

(1) From Lemma 3.5(3), D can be expressed by $D = d\varphi_0(D_1, D_2, D_3)$. Using (3.5), we can show $\sigma_i D\sigma_i X = DX$ on $\mathcal{J}^1$. Then from (1.5.a), (7.1.a)(i) follows. From (3.9.b), showing $\sigma_i\tilde{A}_i(a)\sigma_i X = \tilde{A}_i(a)X$ on $\mathcal{J}^1$, (7.1.a)(ii) follows. Similarly, (7.1.a)(iii) follows.

(2) From (3.9.b) and (3.5), showing $[D, \tilde{A}_i^1(a)]X = \tilde{A}_i^1(D_i a)X$ on $\mathcal{J}^1$, (7.1.b)(i) follows. From (3.9.b), showing $[\tilde{A}_i^1(a), \tilde{A}_{i+1}^1(b)]X = \epsilon(i+2)\tilde{A}_{i+2}^1(\overline{ab})X$ on $\mathcal{J}^1$ and $[\tilde{A}_i^1(a), \tilde{A}_i^1(b)]E_k = 0$ with $k \in \{1, 2, 3\}$, we obtain (7.1.b)(ii)(iii). $\square$

Lemma 7.2. *The following assertions hold.*

(1) ([19, Theorem 2.5.3]). *The Killing form B of $\mathfrak{f}_4^{\mathbb{C}}$ is given by*

$$B(\phi_1, \phi_2) = 3\mathrm{tr}(\phi_1\phi_2) \quad \textit{for } \phi_i \in \mathfrak{f}_4^{\mathbb{C}}. \tag{7.2.a}$$

Especially, the Killing form B of $\mathfrak{f}_{4(-20)} = (\mathfrak{f}_4^{\mathbb{C}})_{\widetilde{\tau\sigma}}$ is given by the restriction $B|(\mathfrak{f}_{4(-20)} \times \mathfrak{f}_{4(-20)})$.

(2) *Let $\phi = d\varphi_0(D_1, D_2, D_3) + \sum_{i=1}^3 \tilde{A}_i^1(a_i)$ where $d\varphi_0(D_1, D_2, D_3) \in \mathfrak{d}_4$ and $a_i \in \mathbf{O}$. Then*

$$B(\phi, \tilde{\sigma}\phi) = -3\left(\sum{}_{i=1}^{3}\left((\sum{}_{j=0}^{7}(D_i e_j|D_i e_j)) + 24(a_i|a_i)\right)\right). \tag{7.2.b}$$

Furthermore, $\tilde{\sigma}$ is a Cartan involution of $\mathfrak{f}_{4(-20)}$.

Proof. (2) By (7.1.a), $\tilde{\sigma}\phi = d\varphi_0(D_1, D_2, D_3) + \sum_{i=1}^3 \epsilon(i)\tilde{A}_i^1(a_i)$. Since $\{E_i, F_i^1(e_j)|\ i = 1, 2, 3,\ j = 0, \cdots, 7\}$ is a basis of $\mathcal{J}^1$, using (7.2.a), $B(\phi, \tilde{\sigma}\phi) = 3\left(\sum{}_{i=1}^{3}(\phi(\tilde{\sigma}\phi)E_i|E_i) + \sum{}_{i=1}^{3}\sum{}_{j=0}^{7}((\phi(\tilde{\sigma}\phi)F_i^1(e_j))_{F_i^1}|e_j)\right)$ and from (3.5) and (3.9.b), we see that $(\phi(\tilde{\sigma}\phi)E_i|E_i) = -2((a_{i+1}|a_{i+1}) + (a_{i+2}|a_{i+2}))$ and $((\phi(\tilde{\sigma}\phi)F_i^1(e_j))_{F_i^1}|e_j) = -(D_i e_j|D_i e_j) - 4(a_i|e_j)^2 - \sum_{j=i+1}^{i+2}(a_j|a_j)$ where indexes $i, i+1, i+2$ are counted modulo 3. Thus (7.2.b) follows. Moreover, the bilinear form $B(\phi_1, \tilde{\sigma}\phi_2)$ ($\phi_1, \phi_2 \in \mathfrak{f}_{4(-20)}$) is negative definite. Hence the result follows. $\square$

Denote $\mathfrak{k} := \{\phi \in \mathfrak{f}_{4(-20)}|\ \tilde{\sigma}\phi = \phi\} = Lie(K)$ and $\mathfrak{p} := \{\phi \in \mathfrak{f}_{4(-20)}|\ \tilde{\sigma}\phi = -\phi\}$. By Lemma 7.2(2),

$$\mathfrak{f}_{4(-20)} = \mathfrak{k} \oplus \mathfrak{p} \quad \text{(Cartan decomposition)}.$$

From (7.1.a)(iii), we see $\tilde{A}_3^1(1) \in \mathfrak{p}$. The abelian subspace $\mathfrak{a}$ of $\mathfrak{p}$ and the linear functional α on $\mathfrak{a}$ are defined as

$$\mathfrak{a} := \{t\tilde{A}_3^1(1)|\ t \in \mathbb{R}\}, \quad \alpha(\tilde{A}_3^1(1)) := 1$$

respectively. Let $\mathfrak{a}_{\mathfrak{p}}$ be a maximal abelian subspace of $\mathfrak{p}$ such that $\mathfrak{a} \subset \mathfrak{a}_{\mathfrak{p}}$, and denote the dual space of $\mathfrak{a}$ (*resp.* $\mathfrak{a}_{\mathfrak{p}}$) as $\mathfrak{a}^*$ (*resp.* $\mathfrak{a}_{\mathfrak{p}}^*$). For

$\lambda \in \mathfrak{a}^*$ (*resp.* $\mathfrak{a}_{\mathfrak{p}}^*$), denote

$$\mathfrak{g}_\lambda := \{\phi \in \mathfrak{f}_{4(-20)} |\ [H,\phi] = \lambda(H)\phi \text{ for all } H \in \mathfrak{a}\}$$
$$(resp.\quad \mathfrak{g}_\lambda(\mathfrak{a}_{\mathfrak{p}}) := \{\phi \in \mathfrak{f}_{4(-20)} |\ [H,\phi] = \lambda(H)\phi \text{ for all } H \in \mathfrak{a}_{\mathfrak{p}}\})$$

and denote

$$\Sigma := \{\lambda \in \mathfrak{a}^* |\ \lambda \neq 0,\ \mathfrak{g}_\lambda \neq \{0\}\}$$
$$(resp.\quad \Sigma(\mathfrak{a}_{\mathfrak{p}}) := \{\lambda \in \mathfrak{a}_{\mathfrak{p}}^* |\ \lambda \neq 0,\ \mathfrak{g}_\lambda(\mathfrak{a}_{\mathfrak{p}}) \neq \{0\}\}).$$

Moreover, the centralizer of $\mathfrak{a}$ of the group K and its Lie algebra as

$$M := Z_K(\mathfrak{a}) = \{k \in K |\ k\tilde{A}_3^1(1)k^{-1} = \tilde{A}_3^1(1)\},$$
$$\mathfrak{m} := Z_{\mathfrak{k}}(\mathfrak{a}) = \{\phi \in \mathfrak{k} \mid [\phi, \tilde{A}_3^1(1)] = 0\}$$

respectively. For all $p \in \mathrm{Im}\mathbf{O}$, the elements $l_p, r_p, t_p \in \mathrm{End}_{\mathbb{R}}(\mathbf{O})$ are defined by

$$l_p x := px, \quad r_p x := xp, \quad t_p x := (l_p + r_p)x = px + xp \qquad \text{for } x \in \mathbf{O}$$

respectively. Because of $\overline{p} = -p$ and (1.1.e), we see $(l_p x|y) = -(x|l_p y)$ so that $D_1 \in \mathfrak{D}_4$. Similarly $l_p, t_p \in \mathfrak{D}_4$. By (1.1.j),

$$l_p(x)y + xr_p(y) = (px)y + x(yp) = p(xy) + (xy)p = \epsilon t_{-p}\epsilon(xy).$$

From Lemma 3.5(3), the element $\delta(p) \in \mathfrak{d}_4$ is defined by

$$\delta(p) := d\varphi_0(l_p, r_p, t_{-p}).$$

For $p \in \mathrm{Im}\mathbf{O}$ and $x \in \mathbf{O}$, denote

$$\mathcal{G}_1(x) := \tilde{A}_1^1(x) + \tilde{A}_2^1(-\overline{x}), \qquad \mathcal{G}_2(p) := \tilde{A}_3^1(-p) - \delta(p),$$
$$\mathcal{G}_{-1}(x) := \tilde{A}_1^1(x) + \tilde{A}_2^1(\overline{x}), \qquad \mathcal{G}_{-2}(p) := \tilde{A}_3^1(p) - \delta(p)$$

and the subspaces $\mathfrak{g}_{\pm 1}$, $\mathfrak{g}_{\pm 2}$ of $\mathfrak{f}_{4(-20)}$ as

$$\mathfrak{g}_1 := \{\mathcal{G}_1(x) |\ x \in \mathbf{O}\}, \qquad \mathfrak{g}_2 := \{\mathcal{G}_2(p) |\ p \in \mathrm{Im}\mathbf{O}\},$$
$$\mathfrak{g}_{-1} := \{\mathcal{G}_{-1}(x) |\ x \in \mathbf{O}\}, \qquad \mathfrak{g}_{-2} := \{\mathcal{G}_{-2}(p) |\ p \in \mathrm{Im}\mathbf{O}\}$$

respectively.

Lemma 7.3. *Let $p \in \mathrm{Im}\mathbf{O}$ and $x \in \mathbf{O}$.*

(1) $\mathfrak{g}_i \subset \mathfrak{g}_{i\alpha}$ *for* $i \in \{\pm 1, \pm 2\}$. *Especially,* $\{\pm\alpha, \pm 2\alpha\} \subset \Sigma$.

(2) $[\mathfrak{g}_{i\alpha}, \mathfrak{g}_{j\alpha}] = \mathfrak{g}_{(i+j)\alpha}$.

Proof. (1) Using (7.1.b)(iii), we calculate that

$$[\tilde{A}_3^1(1), \tilde{A}_1^1(x) + \tilde{A}_2^1(\mp\overline{x})] = \pm(\tilde{A}_1^1(x) + \tilde{A}_2^1(\mp\overline{x})) \quad (resp)$$

with $x \in \mathbf{O}$. Thus $\mathfrak{g}_{\pm 1} \subset \mathfrak{g}_{\pm\alpha}$ (*resp*). Fix $p \in \mathrm{Im}\mathbf{O}$. By (7.1.b)(ii), $[\tilde{A}_3^1(1), \tilde{A}_3^1(p)]E_k = 0$ with $k \in \{1,2,3\}$, and because of (3.9.b), $\overline{p} = -p$ and $4(p|x) - 4(1|x)p = 2(x\overline{p} + p\overline{x}) - 2p(x + \overline{x}) = -2(px + xp)$, we see

$$[\tilde{A}_3^1(1), \tilde{A}_3^1(p)]F_1^1(x) = F_1^1(2px), \quad [\tilde{A}_3^1(1), \tilde{A}_3^1(p)]F_2^1(x) = F_2^1(2xp),$$
$$[\tilde{A}_3^1(1), \tilde{A}_3^1(p)]F_3^1(x) = F_3^1(4(p|x) - 4(1|x)p) = F_3^1(-2(px + xp)).$$

Then from (1.5.a), we see $[\tilde{A}_3^1(1), \tilde{A}_3^1(p)] = 2\delta(p)$ on $\mathcal{J}^1$. Thus, from (7.1.b)(i), we have

$$[\tilde{A}_3^1(1), \tilde{A}_3^1(-p) - \delta(p)] = -2\delta(p) + \tilde{A}_3^1(t_{-p}1) = 2(\tilde{A}_3^1(-p) - \delta(p)),$$
$$[\tilde{A}_3^1(1), \tilde{A}_3^1(p) - \delta(p)] = 2\delta(p) + \tilde{A}_3^1(t_{-p}1) = -2(\tilde{A}_3^1(p) - \delta(p))$$

and so $\mathfrak{g}_{\pm 2} \subset \mathfrak{g}_{\pm 2\alpha}$ (*resp*). Hence (1) follows.

(2) It follows from the Jacobi identity. □

Proposition 7.4. *The following equation hold.*

$$M = \mathrm{B}_3 \cong \mathrm{Spin}(7). \tag{7.4}$$

Proof. From (3.1.b), the subgroup $\mathrm{B}_3 \cong \mathrm{Spin}(7)$ in K is the stabilizer $(\mathrm{F}_{4(-20)})_{E_1, F_3^1(1)}$. Fix $g \in \mathrm{Spin}(7)$. Then g can be expressed by $g = (g_1, \epsilon g_1 \epsilon, g_3)$ such that $g = (g_1, \epsilon g_1 \epsilon, g_3) \in \mathrm{Spin}(8)$ and $g_3 1 = 1$. Put $\phi = \varphi_0(g)\tilde{A}_3^1(1)\varphi_0(g)^{-1}$. Using (3.3) and (3.9.b), we calculate that

$$\phi(-E_1 + E_2) = 2F_3^1(1) = \tilde{A}_3^1(1)(-E_1 + E_2),$$
$$\phi P^- = 2P^- = \tilde{A}_3^1(1)P^-,\ \phi E = 0 = \tilde{A}_3^1(1)E,\ \phi E_3 = 0 = \tilde{A}_3^1(1)E_3,$$
$$\phi F_3^1(p) = 2(g_3^{-1}p|1)(-E_1 + E_2) = 2(p|1)(-E_1 + E_2) = \tilde{A}_3^1(1)F_3^1(p),$$
$$\phi Q^+(x) = Q^+(\overline{x}) = \tilde{A}_3^1(1)Q^+(x),\ \phi Q^-(x) = -Q^-(\overline{x}) = \tilde{A}_3^1(1)Q^-(x)$$

where $x \in \mathbf{O}$ and $p \in \mathrm{Im}\mathbf{O}$. From (1.5.b), we see $\phi = \tilde{A}_3^1(1)$ on $\mathcal{J}^1$. Thus $\varphi_0(g) \in M$ and so $\mathrm{B}_3 \subset M$.

Conversely, take $k \in M \subset K$. Then $k, k^{-1} \in (\mathrm{F}_{4(-20)})_{E_1}$ by (4.14.b) and $k\tilde{A}_3^1(1)k^{-1} = \tilde{A}_3^1(1)$. Now from (3.9.b), we see that $kF_3^1(1) = -k\tilde{A}_3^1(1)E_1 = -k\tilde{A}_3^1(1)k^{-1}E_1 = -\tilde{A}_3^1(1)E_1 = F_3^1(1)$. Thus we obtain $k \in (\mathrm{F}_{4(-20)})_{E_1, F_3^1(1)} = \mathrm{B}_3$ and so $M \subset \mathrm{B}_3$. Hence $M = \mathrm{B}_3$. □

Lemma 7.5. *Let $i \in \{\pm 1, \pm 2\}$. The following equations hold.*

$$\text{(i)}\quad \mathfrak{g}_i = \mathfrak{g}_{i\alpha}, \quad \text{(ii)}\quad \mathfrak{a}_{\mathfrak{p}} = \mathfrak{a}. \tag{7.5.a}$$

$$\Sigma(\mathfrak{a}_{\mathfrak{p}}) = \Sigma = \{\pm\alpha, \pm 2\alpha\}. \tag{7.5.b}$$

$$\mathfrak{f}_{4(-20)} = \mathfrak{g}_{-2\alpha} \oplus \mathfrak{g}_{-\alpha} \oplus \mathfrak{a} \oplus \mathfrak{m} \oplus \mathfrak{g}_{\alpha} \oplus \mathfrak{g}_{2\alpha}. \tag{7.5.c}$$

Proof. From the definitions of $\mathfrak{m}$ and $\mathfrak{a}$, we see $\mathfrak{m} \subset \mathfrak{g}_0 \cap \mathfrak{k}$ and $\mathfrak{a} \subset \mathfrak{g}_0 \cap \mathfrak{p}$. Then from Lemma 7.3(1), we see that $\mathfrak{g}_{-2} + \mathfrak{g}_{-1} + \mathfrak{a} + \mathfrak{m} + \mathfrak{g}_1 + \mathfrak{g}_2$ is a direct sum $\mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{a} \oplus \mathfrak{m} \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2$ and that

$$\mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{a} \oplus \mathfrak{m} \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2 \subset \mathfrak{g}_{-2\alpha} \oplus \mathfrak{g}_{-\alpha} \oplus \mathfrak{a} \oplus \mathfrak{m} \oplus \mathfrak{g}_{\alpha} \oplus \mathfrak{g}_{2\alpha} \subset \mathfrak{f}_{4(-20)}.$$

Now $\dim \mathfrak{a} = 1$, $\dim \mathfrak{g}_2 = \dim \mathfrak{g}_{-2} = \dim \mathbf{O} = 8$, $\dim \mathfrak{g}_1 = \dim \mathfrak{g}_{-1} = \dim(\mathrm{Im}\mathbf{O}) = 7$ and by (7.4), $\dim \mathfrak{m} = \dim(\mathfrak{so}(7)) = 21$ where $\mathfrak{so}(7) = \mathit{Lie}(\mathrm{SO}(7))$. By (3.9.a), $\dim \mathfrak{f}_{4(-20)} = 52$. Thus

$$\dim(\mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{a} \oplus \mathfrak{m} \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2) = 52 = \dim \mathfrak{f}_{4(-20)}$$

and it is clear that

$$\mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{a} \oplus \mathfrak{m} \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2 = \mathfrak{g}_{-2\alpha} \oplus \mathfrak{g}_{-\alpha} \oplus \mathfrak{a} \oplus \mathfrak{m} \oplus \mathfrak{g}_\alpha \oplus \mathfrak{g}_{2\alpha} = \mathfrak{f}_{4(-20)}$$

and that $\mathfrak{g}_i = \mathfrak{g}_{i\alpha}$ for $i \in \{\pm 1, \pm 2\}$. Because of $\mathfrak{a} \oplus \mathfrak{m} \subset \mathfrak{g}_0$, the decomposition $\mathfrak{f}_{4(-20)} = \mathfrak{g}_{-2\alpha} \oplus \mathfrak{g}_{-\alpha} \oplus (\mathfrak{a} \oplus \mathfrak{m}) \oplus \mathfrak{g}_\alpha \oplus \mathfrak{g}_{2\alpha}$ implies the eigendecomposition of $\mathrm{ad}(\tilde{A}_3^1(1))$ and the root space decomposition of $(\mathfrak{f}_{4(-20)}, \mathfrak{a})$ (cf. [10, Ch V]). Thus $\Sigma = \{\pm\alpha, \pm 2\alpha\}$ and $\mathfrak{g}_0 = \mathfrak{a} \oplus \mathfrak{m}$. Because of $\mathfrak{a} \subset \mathfrak{a}_\mathfrak{p} \subset \mathfrak{g}_0 \cap \mathfrak{p} = \mathfrak{a}$, we have $\mathfrak{a}_\mathfrak{p} = \mathfrak{a}$ and $\Sigma(\mathfrak{a}_\mathfrak{p}) = \Sigma = \{\pm\alpha, \pm 2\alpha\}$. $\square$

The nilpotent subalgebras $\mathfrak{n}^\pm$ are defined by

$$\mathfrak{n}^+ := \mathfrak{g}_{2\alpha} \oplus \mathfrak{g}_\alpha = \{\mathcal{G}_2(p) + \mathcal{G}_1(x) |\ p \in \mathrm{Im}\mathbf{O}, x \in \mathbf{O}\},$$
$$\mathfrak{n}^- := \mathfrak{g}_{-2\alpha} \oplus \mathfrak{g}_{-\alpha} = \{\mathcal{G}_{-2}(p) + \mathcal{G}_{-1}(x) |\ p \in \mathrm{Im}\mathbf{O}, x \in \mathbf{O}\}$$

respectively. In fact, by Lemma 7.3(2) and (7.5.b),

$$[\mathfrak{n}^+, [\mathfrak{n}^+, \mathfrak{n}^+]] = 0, \qquad [\mathfrak{n}^-, [\mathfrak{n}^-, \mathfrak{n}^-]] = 0$$

and the nilpotent subgroups $N^\pm$ of $\mathrm{F}_{4(-20)}$ are defined as

$$N^+ := \exp \mathfrak{n}^+ = \{\exp(\mathcal{G}_2(p) + \mathcal{G}_1(x)) \mid p \in \mathrm{Im}\mathbf{O}, x \in \mathbf{O}\},$$
$$N^- := \exp \mathfrak{n}^- = \{\exp(\mathcal{G}_{-2}(p) + \mathcal{G}_{-1}(x)) \mid p \in \mathrm{Im}\mathbf{O}, x \in \mathbf{O}\}$$

respectively.

Lemma 7.6. *Let $x \in \mathbf{O}$ and $p \in \mathrm{Im}\mathbf{O}$.*

$$\exp \mathcal{G}_2(p) \exp \mathcal{G}_1(x) = \exp(\mathcal{G}_2(p) + \mathcal{G}_1(x)) = \exp \mathcal{G}_1(x) \exp \mathcal{G}_2(p). \tag{7.6}$$

Proof. By Lemma 7.3(2) and (7.5.b), $[\mathfrak{g}_\alpha, \mathfrak{g}_{2\alpha}] = [\mathfrak{g}_{2\alpha}, \mathfrak{g}_\alpha] = 0$. Hence (7.6) follows. $\square$

Denote $T(x, y, z) := (x\overline{y})z - (z\overline{y})x$ for $x, y, z \in \mathbf{O}$ (cf. [6, (6.55)]).

Lemma 7.7. ([6, Lemma 6.56]). *$T(x, y, z)$ is alternating on $\mathbf{O}$: $T(x, x, z) = T(z, x, x) = T(x, z, x) = 0$ for all $x, z \in \mathbf{O}$. Especially, for all $x_1, x_2, x_3 \in \mathbf{O}$,*

$$T(x_1, x_2, x_3) = T(x_i, x_{i+1}, x_{i+2}) = -T(x_i, x_{i+2}, x_{i+1}) \tag{7.7}$$

where $i \in \{1, 2, 3\}$ and the indexes $i, i+1, i+2$ are counted modulo 3.

Proof. It follows from (1.1.h). $\square$

Lemma 7.8. *Let $p, q \in \mathrm{Im}\mathbf{O}$ and $x, y \in \mathbf{O}$. Then*

$$[\mathcal{G}_2(p) + \mathcal{G}_1(x),\ \mathcal{G}_2(q) + \mathcal{G}_1(y)] = \mathcal{G}_2(2\mathrm{Im}(x\overline{y})). \tag{7.8}$$

Proof. First, by Lemma 7.3(2) and (7.5.b),

$$[\mathcal{G}_2(p), \mathcal{G}_2(q)] = [\mathcal{G}_2(p), \mathcal{G}_1(x)] = 0.$$

Second, we show $[\mathcal{G}_1(x), \mathcal{G}_1(y)] = \mathcal{G}_2(2\mathrm{Im}(x\overline{y}))$. Put $f = [\mathcal{G}_1(x), \mathcal{G}_1(y)] - \mathcal{G}_2(2\mathrm{Im}(x\overline{y}))$. From (7.1.b), we calculate that

$$[\mathcal{G}_1(x), \mathcal{G}_1(y)] = ([\tilde{A}_1^1(x), \tilde{A}_1^1(y)] + [\tilde{A}_2^1(\overline{x}), \tilde{A}_2^1(\overline{x})]) - \tilde{A}_3^1(2\mathrm{Im}(x\overline{y})),$$
$$\mathcal{G}_2(2\mathrm{Im}(x\overline{y})) = \tilde{A}_3^1(-2\mathrm{Im}(x\overline{y})) - \delta(2\mathrm{Im}(x\overline{y})).$$

Then $f = [\tilde{A}_1^1(\overline{x}), \tilde{A}_1^1(\overline{y})] + [\tilde{A}_2^1(x), \tilde{A}_2^1(y)] + \delta(2\mathrm{Im}(x\overline{y}))$. By (7.1.b)(ii), $f \in \mathfrak{d}_4$ and from Lemma 3.5(3), we can write $f = d\varphi_0(D_1, D_2, D_3)$ where $(D_1, D_2, D_3) \in (\mathfrak{D}_4)^3$ satisfying the infinitesimal triality. Fix $z \in \mathbf{O}$. Then

$$\begin{aligned} F_1^1(D_1 z) &= d\varphi_0(D_1, D_2, D_3) F_1^1(z) = f F_1^1(z) \\ &= ([\tilde{A}_1^1(x), \tilde{A}_1^1(y)] + [\tilde{A}_2^1(\overline{x}), \tilde{A}_2^1(\overline{y})] + \delta(2\mathrm{Im}(x\overline{y}))) F_1^1(z) \end{aligned}$$

and using (3.9.b), (1.1.g), and (7.7), we see that

$$\begin{aligned} D_1 z &= -4(y|z)x + 4(x|z)y + (z\overline{y})x - (z\overline{x})y + (x\overline{y})z - (y\overline{x})z \\ &= -2(y\overline{z})z - 2(z\overline{y})x + 2(x\overline{z})y + 2(z\overline{x})y \\ &\quad + (z\overline{y})x - (z\overline{x})y + (x\overline{y})z - (y\overline{x})z \\ &= (z\overline{x})y - (y\overline{x})z + (x\overline{y})z - (z\overline{y})x + 2(x\overline{z})y - 2(y\overline{z})x \\ &= T(z, x, y) + T(x, y, z) + 2T(x, z, y) \\ &= T(x, y, z) + T(x, y, z) - 2T(x, y, z) = 0. \end{aligned}$$

Then $D_1 = 0$ and from Lemma 3.5(1), we see $D_2 = D_3 = 0$. Thus $f = 0$ and so $[\mathcal{G}_1(x), \mathcal{G}_1(y)] = \mathcal{G}_2(2\mathrm{Im}(x\overline{y}))$. Hence the result follows. □

Lemma 7.9. *There exists a neighborhood U of 0 in $\mathrm{Im}\mathbf{O} \times \mathbf{O}$ such that*

$$\begin{aligned} &\exp(\mathcal{G}_2(p) + \mathcal{G}_1(x)) \exp(\mathcal{G}_2(q) + \mathcal{G}_1(y)) \\ &= \exp(\mathcal{G}_2(p + q + \mathrm{Im}(x\overline{y})) + \mathcal{G}_1(x + y)) \end{aligned} \tag{7.9}$$

for all $(p, x), (q, y) \in U$.

Proof. Using Campbell-Hausdorff-Dynkin formulas (cf. [3, Theorem 3.4.4]), there exists a neighborhood U_1 of 0 in $\mathrm{End}_{\mathbb{R}}(\mathcal{J}^1)$ such that for all $X, Y \in U_1$,

$$\begin{aligned} \exp X \exp Y = \exp \bigl(&X + Y + 2^{-1}[X, Y] + 12^{-1}[X, [X, Y]] \\ &+ 12^{-1}[Y, [Y, X]] + (\text{terms of degree} \geq 4)\bigr). \end{aligned}$$

Because of $[\mathfrak{n}^+, [\mathfrak{n}^+, \mathfrak{n}^+]] = 0$, we see

$$\exp X \exp Y = \exp \bigl(X + Y + 2^{-1}[X, Y]\bigr)$$

for all $X, Y \in \mathfrak{n}^+ \cap U_1$. Then by (7.8), there exists a neighborhood U of 0 in $\mathrm{Im}\mathbf{O} \times \mathbf{O}$ such that for all $(p,x),(q,y) \in U$,

$$\begin{aligned}
&\exp(\mathcal{G}_2(p)+\mathcal{G}_1(x))\exp(\mathcal{G}_2(q)+\mathcal{G}_1(y)))\\
&=\exp\big(\mathcal{G}_2(p+q)+\mathcal{G}_1(x+y)+2^{-1}[\mathcal{G}_2(p)+\mathcal{G}_1(x),\mathcal{G}_2(q)+\mathcal{G}_1(y)]\big)\\
&=\exp(\mathcal{G}_2(p+q+\mathrm{Im}(x\overline{y}))+\mathcal{G}_1(x+y)).
\end{aligned}$$

□

Lemma 7.10. *Let $p,q \in \mathrm{Im}\mathbf{O}$ and $x,y \in \mathbf{O}$.*

(7.10.a)

$$\begin{cases}
\text{(i)}\quad \mathcal{G}_2(p)(-E_1+E_2) = -2F_3^1(p),\\
\text{(ii)}\quad \mathcal{G}_2(p)P^- = 0, \quad \text{(iii) } \mathcal{G}_2(p)E = 0, \quad \text{(iv) } \mathcal{G}_2(p)E_3 = 0,\\
\text{(v)}\quad \mathcal{G}_2(p)F_3^1(q) = -2(p|q)P^-,\\
\text{(vi)}\quad \mathcal{G}_2(p)Q^+(y) = 0, \quad \text{(vii) } \mathcal{G}_2(p)Q^-(y) = -2Q^+(py).
\end{cases}$$

(7.10.b)

$$\begin{cases}
\text{(i)}\quad \mathcal{G}_1(x)(-E_1+E_2) = -Q^-(x),\\
\text{(ii)}\quad \mathcal{G}_1(x)P^- = 0, \quad \text{(iii) } \mathcal{G}_1(x)E = 0, \quad \text{(iv) } \mathcal{G}_1(x)E_3 = Q^+(x),\\
\text{(v)}\quad \mathcal{G}_1(x)F_3^1(q) = -Q^+(qx), \quad \text{(vi) } \mathcal{G}_1(x)Q^+(y) = 2(x|y)P^-,\\
\text{(vii)}\quad \mathcal{G}_1(x)Q^-(y) = 2(x|y)(E-3E_3)+F_3^1(2\mathrm{Im}(x\overline{y})).
\end{cases}$$

Proof. Using (3.9.b) and the definition of $\delta(p)$, it follows from direct calculations. □

Lemma 7.11. *Let $p,q \in \mathrm{Im}\mathbf{O}$ and $x,y \in \mathbf{O}$.*

(7.11.a)

$$\begin{cases}
\text{(i)}\quad \exp\mathcal{G}_2(p)(-E_1+E_2) = (-E_1+E_2)-2F_3^1(p)+2(p|p)P^-,\\
\text{(ii)}\quad \exp\mathcal{G}_2(p)P^- = P^-, \quad \text{(iii) } \exp\mathcal{G}_2(p)E = E,\\
\text{(iv)}\quad \exp\mathcal{G}_2(p)E_3 = E_3,\\
\text{(v)}\quad \exp\mathcal{G}_2(p)F_3^1(q) = F_3^1(q)-2(p|q)P^-,\\
\text{(vi)}\quad \exp\mathcal{G}_2(p)Q^+(y) = Q^+(y),\\
\text{(vii)}\quad \exp\mathcal{G}_2(p)Q^-(y) = Q^-(y)-2Q^+(py).
\end{cases}$$

(7.11.b)

$$\begin{cases}
\text{(i)}\quad \exp\mathcal{G}_1(x)(-E_1+E_2) = (-E_1+E_2)-Q^-(x)\\
\qquad -(x|x)(E-3E_3)+(x|x)Q^+(x)+2^{-1}(x|x)^2P^-,\\
\text{(ii)}\quad \exp\mathcal{G}_1(x)P^- = P^-, \quad \text{(iii) } \exp\mathcal{G}_1(x)E = E,\\
\text{(iv)}\quad \exp\mathcal{G}_1(x)E_3 = E_3+Q^+(x)+(x|x)P^-,\\
\text{(v)}\quad \exp\mathcal{G}_1(x)F_3^1(q) = F_3^1(q)-Q^+(qx),\\
\text{(vi)}\quad \exp\mathcal{G}_1(x)Q^+(y) = Q^+(y)+2(x|y)P^-,\\
\text{(vii)}\quad \exp\mathcal{G}_1(x)Q^-(y) = Q^-(y)+2(x|y)(E-3E_3)\\
\qquad +F_3^1(2\mathrm{Im}(x\overline{y}))-Q^+(3(x|y)x+\mathrm{Im}(x\overline{y})x)-2(x|y)(x|x)P^-.
\end{cases}$$

Proof. Using (7.10.a) and (7.10.b), it follows from direct calculations. □

Lemma 7.12. *Let* $p \in \mathrm{Im}\mathbf{O}$, $x \in \mathbf{O}$ *and* $X \in \mathfrak{n}^+$. *Then* $\mathcal{G}_2(p)^3 = 0$ *and* $\mathcal{G}_1(x)^5 = 0$ *Especially,*
(7.12)
$$\exp \mathcal{G}_2(p) = \sum\nolimits_{n=0}^{2} (n!)^{-1} \mathcal{G}_2(p)^n, \quad \exp \mathcal{G}_1(x) = \sum\nolimits_{n=0}^{4} (n!)^{-1} \mathcal{G}_1(x)^n.$$

Proof. Let $S = \{-E_1 + E_2,\ P^-,\ E,\ E_3,\ F_3^1(p),\ Q^+(x),\ Q^-(y) |\ p \in \mathrm{Im}\mathbf{O},\ x, y \in \mathbf{O}\}$. From (7.10), we see that $\mathcal{G}_2(p)^3 Y = 0$ and $\mathcal{G}_1(x)^5 Y = 0$ for all $Y \in S$. Thus it follows from (1.5.b) that $\mathcal{G}_2(p)^3 = 0$ and $\mathcal{G}_1(x)^5 = 0$ on $\mathcal{J}^1$. □

8. The stabilizers of semidirect product group type.

Consider $\mathrm{Spin}(7) \times \mathrm{Im}\mathbf{O} \times \mathbf{O}$ in which multiplication is defined by
$$(g, p, x)(h, q, y) := (gh, p + g_3 q + \mathrm{Im}(x\overline{(g_1 y)}), x + g_1 y)$$
where $p, q \in \mathrm{Im}\mathbf{O}$, $x, y \in \mathbf{O}$ and $g = (g_1, g_2, g_3), h \in \mathrm{Spin}(7)$. By Lemma 2.3(1), $g_3 q \in \mathrm{Im}\mathbf{O}$ and it is clear that the multiplication is closed. Denote $G := \mathrm{Spin}(7) \times \mathrm{Im}\mathbf{O} \times \mathbf{O}$, $G_0 := \{(g, 0, 0) |\ g \in \mathrm{Spin}(7)\}$ and define the subset $\mathrm{H}_{\mathrm{Im}\mathbf{O},\mathbf{O}}$ of G by
$$\mathrm{H}_{\mathrm{Im}\mathbf{O},\mathbf{O}} := \{(1, p, x) |\ p \in \mathrm{Im}\mathbf{O}, x \in \mathbf{O}\}.$$
Then, for all $(1, p, x), (1, q, y) \in \mathrm{H}_{\mathrm{Im}\mathbf{O},\mathbf{O}}$,
$$(1, p, x)(1, q, y) = (1, p + q + \mathrm{Im}(x\overline{y}), x + y).$$
$\mathrm{H}_{\mathrm{Im}\mathbf{O},\mathbf{O}}$ has a group structure from the following lemma and is called the *Heisenberg group of* $\mathbf{O}$.

Lemma 8.1. (1) *G is a group with respect to the multiplication.*
(2) *G_0 and $\mathrm{H}_{\mathrm{Im}\mathbf{O},\mathbf{O}}$ are subgroups of G; $G_0 \cong \mathrm{Spin}(7)$.*
(3) *G is the semi-direct product $G_0 \ltimes \mathrm{H}_{\mathrm{Im}\mathbf{O},\mathbf{O}}$.*

Proof. (1) Let $g = (g_1, g_2, g_3), h = (h_1, h_2, h_3), f \in \mathrm{Spin}(7)$, $p, q, r \in \mathrm{Im}\mathbf{O}$ and $x, y, z \in \mathbf{O}$. Because of $g_3 1 = 1$ and (2.3.c)(ii), we see $g_3(\mathrm{Im}(y\overline{(h_1 z)}) = \mathrm{Im}((g_1 y)\overline{(g_1 h_1 z)})$. Then
$$\begin{aligned}
&((g, p, x)(h, q, y))(f, r, z) \\
&= (ghf, p + g_3 q + g_3 h_3 r + \mathrm{Im}(x\overline{(g_1 y)}) \\
&\quad + \mathrm{Im}(x\overline{(g_1 h_1 z)}) + \mathrm{Im}((g_1 y)\overline{(g_1 h_1 z)}), x + g_1 y + g_1 h_1 z) \\
&= (ghf, p + g_3 q + g_3 h_3 r + \mathrm{Im}(x\overline{(g_1 y)}) \\
&\quad + \mathrm{Im}(x\overline{(g_1 h_1 z)}) + g_3(\mathrm{Im}(y\overline{(h_1 z)})), x + g_1 y + g_1 h_1 z) \\
&= (g, p, x)((h, q, y)(f, r, z)).
\end{aligned}$$

Thus the associativity hold. The identity element is $(1, 0, 0)$ and the inverse element of (g, p, x) is $(g^{-1}, -g_3^{-1}p, -g_1^{-1}x)$. Hence (1) follows.

(2) Because $(g, 0, 0)^{-1}(h, 0, 0) = (g^{-1}h, 0, 0)$ and $(1, p, x)^{-1}(1, q, y) = (1, *, *)$, we see that G_0 and $\mathrm{H}_{\mathrm{Im}\mathbf{O},\mathbf{O}}$ are subgroups of G and obviously, $G_0 \cong \mathrm{Spin}(7)$. Hence (2) follows.

(3) Let $(g, p, x) \in \mathrm{H}_{\mathrm{Im}\mathbf{O},\mathbf{O}}$ where $g = (g_1, g_2, g_3) \in \mathrm{Spin}(7)$. Then $G = G_0\mathrm{H}_{\mathrm{Im}\mathbf{O},\mathbf{O}}$ follows from $(g, p, x) = (g, 0, 0)(1, g_3^{-1}p, g_1^{-1}x)$. Obviously $G_0 \cap \mathrm{H}_{\mathrm{Im}\mathbf{O},\mathbf{O}} = \{(1, 0, 0)\}$. Because of $(g, p, x)(1, q, y)(g, p, x)^{-1} = (1, *, *)$, we see $(g, p, x)(\mathrm{H}_{\mathrm{Im}\mathbf{O},\mathbf{O}})(g, p, x)^{-1} \subset \mathrm{H}_{\mathrm{Im}\mathbf{O},\mathbf{O}}$. Thus $\mathrm{H}_{\mathrm{Im}\mathbf{O},\mathbf{O}}$ is a normal subgroup of G. Hence $G = G_0 \ltimes \mathrm{H}_{\mathrm{Im}\mathbf{O},\mathbf{O}}$. □

Hereafter, we identify G_0 with $\mathrm{Spin}(7)$. Then we can write $G = \mathrm{Spin}(7) \ltimes \mathrm{H}_{\mathrm{Im}\mathbf{O},\mathbf{O}}$ and use the notation $\mathrm{Spin}(7) \ltimes \mathrm{H}_{\mathrm{Im}\mathbf{O},\mathbf{O}}$. Next, we denote $G' := \{(g, p, 0) |\ g \in \mathrm{Spin}(7), p \in \mathrm{Im}\mathbf{O}\}$ and $N' := \{(1, p, 0) |\ p \in \mathrm{Im}\mathbf{O}\} \subset G'$. Moreover, we denote $G'' := \{(g, p, x) |\ g \in \mathrm{G}_2,\ p, x \in \mathrm{Im}\mathbf{O}\}$ and $G''_0 := \{(g, 0, 0) \mid g \in \mathrm{G}_2\} \subset G''$ and define

$$\mathrm{H}_{\mathrm{Im}\mathbf{O},\mathrm{Im}\mathbf{O}} := \{(1, p, q) \mid p, q \in \mathrm{Im}\mathbf{O}\} \subset G''.$$

Easily, we can prove the following lemma.

Lemma 8.2. (1) *G' and G'' are subgroups of* $\mathrm{Spin}(7) \ltimes \mathrm{H}_{\mathrm{Im}\mathbf{O},\mathbf{O}}$.

(2) $\mathrm{Spin}(7)$ *and N' are subgroups of G'*; $N' \cong \mathrm{Im}\mathbf{O}$.

(3) G''_0 *and* $\mathrm{H}_{\mathrm{Im}\mathbf{O},\mathrm{Im}\mathbf{O}}$ *are subgroups of G''*; $G''_0 \cong \mathrm{G}_2$.

(4) *G' is the semi-direct product* $\mathrm{Spin}(7) \ltimes N$.

(5) *G'' is the semi-direct product* $G''_0 \ltimes \mathrm{H}_{\mathrm{Im}\mathbf{O},\mathrm{Im}\mathbf{O}}$.

Hereafter, we identify N' with $\mathrm{Im}\mathbf{O}$ and G''_0 with G_2. Then we can write $G' = \mathrm{Spin}(7) \ltimes \mathrm{Im}\mathbf{O}$ and $G'' = \mathrm{G}_2 \ltimes \mathrm{H}_{\mathrm{Im}\mathbf{O},\mathrm{Im}\mathbf{O}}$, respectively, and use the notations $\mathrm{Spin}(7) \ltimes \mathrm{Im}\mathbf{O}$ and $\mathrm{G}_2 \ltimes \mathrm{H}_{\mathrm{Im}\mathbf{O},\mathrm{Im}\mathbf{O}}$. The map $\varphi : \mathrm{Spin}(7) \ltimes \mathrm{H}_{\mathrm{Im}\mathbf{O},\mathbf{O}} \to (\mathrm{F}_{4(-20)})_{P^-}$ is defined by

$$\begin{aligned}\varphi(g, p, x) &:= \exp(\mathcal{G}_2(p) + \mathcal{G}_1(x))\varphi_0(g)\\ &= \exp \mathcal{G}_2(p) \exp \mathcal{G}_1(x)\varphi_0(g) = \exp \mathcal{G}_1(x) \exp \mathcal{G}_2(p)\varphi_0(g)\end{aligned}$$

for $(g, p, x) \in \mathrm{Spin}(7) \ltimes \mathrm{H}_{\mathrm{Im}\mathbf{O},\mathbf{O}}$ (see (7.6)). From (3.3), (7.11.a) and (7.11.b), we see $\varphi(g, p, x)P^- = P^-$. So φ is well-defined.

Lemma 8.3. *Let $g = (g_1, g_2, g_3) \in \mathrm{Spin}(7)$, $p, q \in \mathrm{Im}\mathbf{O}$ and $x, y \in \mathbf{O}$.*

$$\varphi_0(g) \exp(\mathcal{G}_2(p) + \mathcal{G}_1(x))\varphi_0(g)^{-1} = \exp(\mathcal{G}_2(g_3p) + \mathcal{G}_1(g_1x)). \tag{8.3}$$

Proof. Put $S = \{-E_1 + E_2,\ P^-,\ E,\ E_3,\ F_3^1(q),\ Q^+(y),\ Q^-(z) |\ q \in \mathrm{Im}\mathbf{O},\ y, z \in \mathbf{O}\}$ and $A = \varphi_0(g)$. Because of $A \exp(\mathcal{G}_2(p)+\mathcal{G}_1(x))A^{-1} = \exp(A(\mathcal{G}_2(p) + \mathcal{G}_1(x))A^{-1})$ and (1.5.b), it is enough to show that

$$\text{(i) } A\mathcal{G}_2(p)A^{-1}X = \mathcal{G}_2(g_3p)X, \quad \text{(ii) } A\mathcal{G}_1(x)A^{-1}X = \mathcal{G}_2(g_1x)X$$

for all $X \in S$.

(Step 1) We show (i) by using (3.3) and (7.10.a).

Case $X = P^-, E, E_3, Q^+(y)$. Then $A\mathcal{G}_2(p)A^{-1}X = 0 = \mathcal{G}_2(g_3p)X$.

Case $X = -E_1 + E_2$. Then $A\mathcal{G}_2(p)A^{-1}(-E_1 + E_2) = -2F_3^1(g_3p) = \mathcal{G}_2(g_3p)(-E_1 + E_2)$.

Case $X = F_3^1(q)$. Because $(g_1, g_2, g_3) \in \mathrm{Spin}(7)$, $A\mathcal{G}_2(p)A^{-1}F_3^1(q) = -2(p|g_3^{-1}q)P^- = -2(g_3p|q)P^- = \mathcal{G}_2(g_3p)F_3^1(q)$.

Case $X = Q^-(z)$. From (2.3.c)(iii), we see that $A\mathcal{G}_2(p)A^{-1}Q^-(z) = -2Q^+(g_1(p(g_1^{-1}z))) = -2Q^+((g_3p)z) = \mathcal{G}_2(g_3p)Q^-(z)$.

Thus (i) follows.

(Step 2) We show by (ii) using (3.3) and (7.10.b).

Case $X = E, P^-$. Then $A\mathcal{G}_1(x)A^{-1}X = 0 = \mathcal{G}_1(g_2x)X$.

Case $X = -E_1 + E_2$. Then $A\mathcal{G}_1(x)A^{-1}(-E_1 + E_2) = -Q^-(g_1x) = \mathcal{G}_1(g_1x)(-E_1 + E_2)$.

Case $X = E_3$. Then $A\mathcal{G}_1(x)A^{-1}E_3 = Q^+(g_1x) = \mathcal{G}_1(g_1x)E_3$.

Case $X = Q^+(y)$. Because $(g_1, g_2, g_3) \in \mathrm{Spin}(7)$, $A\mathcal{G}_1(x)A^{-1}Q^+(y) = -2(x|g_1^{-1}y)P^- = -2(g_1x|y)P^- = \mathcal{G}_2(g_1p)Q^+(y)$.

Case $X = F_3^1(q)$. From (2.3.c)(iii), we see that $A\mathcal{G}_1(x)A^{-1}F_3^1(q) = F_3^1(g_1((g_3^{-1}q)x)) = F_3^1(q(g_1x)) = \mathcal{G}_1(g_1x)F_3^1(q)$.

Case $X = Q^-(z)$. Because of $(g_1, g_2, g_3) \in \mathrm{Spin}(7)$ and (2.3.c)(ii), $A\exp\mathcal{G}_1(x)A^{-1}Q^-(z) = 2(x|g_1^{-1}z)(E - 3E_3) + F_3^1(2g_3\mathrm{Im}(x\overline{(g_1^{-1}z)})) = 2(g_1x|z)(E - 3E_3) + F_3^1(2(g_1x)\overline{z}) = \mathcal{G}_1(g_1x)Q^-(z)$.

Thus (ii) follows. Hence the result follows. □

Lemma 8.4. *Let* $g = (g_1, g_2, g_3) \in \mathrm{Spin}(7)$, $p, q \in \mathrm{Im}\mathbf{O}$ *and* $x, y \in \mathbf{O}$.

$$\begin{aligned}(8.4)\qquad &\exp(\mathcal{G}_2(p) + \mathcal{G}_1(x))\exp(\mathcal{G}_2(q) + \mathcal{G}_1(y)) \\ &= \exp(\mathcal{G}_2(p + q + \mathrm{Im}(x\overline{y})) + \mathcal{G}_1(x + y)).\end{aligned}$$

Proof. Put $f(p, x, q, y) \in \mathrm{End}_{\mathbb{R}}(\mathcal{J}^1)$ as

$$\begin{aligned}f(p, x, q, y) = &\exp(\mathcal{G}_2(p) + \mathcal{G}_1(x))\exp(\mathcal{G}_2(q) + \mathcal{G}_1(y)) \\ &- \exp(\mathcal{G}_2(p + q + \mathrm{Im}(x\overline{y})) + \mathcal{G}_1(x + y)).\end{aligned}$$

Let p_i, q_i, x_i, y_i be variables defined as $p = \sum_{i=1}^7 p_ie_i$, $q = \sum_{i=1}^7 q_ie_i$, $x = \sum_{i=0}^7 x_ie_i$ and $y = \sum_{i=0}^7 y_ie_i$. Fix $X, Y \in \mathcal{J}^1$. Put the function $F_{X,Y}(p, x, q, y) = (f(p, x, q, y)X|Y)$. From (7.6) and (7.12), we see

$$\begin{aligned}f(p, x, q, y) = &(\textstyle\sum_{i=0}^2(i!)^{-1}\mathcal{G}_2(p)^i)(\sum_{i=0}^4(i!)^{-1}\mathcal{G}_1(x)^i) \\ &(\textstyle\sum_{i=0}^2(i!)^{-1}\mathcal{G}_2(q)^i)(\sum_{i=0}^4(i!)^{-1}\mathcal{G}_1(y)^i) \\ &-(\textstyle\sum_{i=0}^2(i!)^{-1}\mathcal{G}_2(p + q + \mathrm{Im}(x\overline{y}))^i)(\sum_{i=0}^4(i!)^{-1}\mathcal{G}_1(x + y)^i)\end{aligned}$$

and it is clear that $F_{X,Y}(p, x, q, y)$ is a polynomial function. Now from Lemma 7.9, there exists a neighborhood U of 0 in $(\mathrm{Im}\mathbf{O} \times \mathbf{O})^2$ such that $f(p, x, q, y) = 0$ for all $(p, x, q, y) \in U$. Then $F_{X,Y}(p, x, q, y) = 0$ for all $(p, x, q, y) \in U$. Since $F_{X,Y}(p, x, q, y)$ is a polynomial function, $F_{X,Y}(p, x, q, y) = 0$ for all $(p, x, q, y) \in (\mathrm{Im}\mathbf{O}\times\mathbf{O})^2$. Moving $X, Y \in \mathcal{J}^1$, we obtain $f(p, x, q, y) = 0$. Hence the result follows. □

Lemma 8.5. *φ is a group homomorphism. Furthermore,*

$$\varphi(\mathrm{Spin}(7) \ltimes \mathrm{H}_{\mathrm{Im}\mathbf{O},\mathbf{O}}) = N^+M \subset (\mathrm{F}_{4(-20)})_{P^-}. \tag{8.5}$$

Proof. Let $F(p_0, x_0) = \exp(\mathcal{G}_2(p_0)+\mathcal{G}_1(x_0))$ for $(p_0, x_0) \in \mathrm{H}_{\mathrm{Im}\mathbf{O},\mathbf{O}}$. From (8.3), (8.4) and Lemma 3.2(2), we see

$$\begin{aligned}
&\varphi(g, p, x)\varphi(h, q, y) = F(p, x)(\varphi_0(g)F(q, y)\varphi_0(g)^{-1})\varphi_0(gh)\\
=&F(p, x)F(g_3q, g_1y)\varphi_0(gh) = F(p + g_3q + \mathrm{Im}(x\overline{(g_1y)}), x + g_1y)\varphi_0(gh)\\
=&\varphi(gh, p + g_3q + \mathrm{Im}(x\overline{(g_1y)}), x + g_1y).
\end{aligned}$$

Thus φ is a group homomorphism. Moreover, (8.5) follows from the definition of φ and (7.4). $\square$

Let V be a $\mathbb{R}$-linear space and N a nilpotent subgroup of $\mathrm{GL}_{\mathbb{R}}(V)$. For $v \in V$, the subset $Orb_N(v)$ of V is called a *parabolic type plane.* Denote nilpotent subgroups N_1 and N_2 in $N^+ = \varphi(\mathrm{H}_{\mathrm{Im}\mathbf{O},\mathbf{O}})$ as

$$N_1 := \varphi(\mathrm{Im}\mathbf{O}), \quad N_2 := \varphi(\mathrm{H}_{\mathrm{Im}\mathbf{O},\mathrm{Im}\mathbf{O}})$$

respectively, and the subsets $\mathcal{P}_{E_3,P^-}$, $\mathcal{P}_{P^-}$ and $\mathcal{P}_{Q^+(1)}$ of $\mathcal{J}^1$ as

$$\begin{aligned}
\mathcal{P}_{E_3,P^-} &:= \{X \in \mathcal{J}^1_{L^\times(2E_3),-1} \mid P^- \times X = -E_3,\ Q(X) = 1\},\\
\mathcal{P}_{P^-} &:= \{X \in \mathcal{J}^1 \mid P^- \times X = -2^{-1}P^-,\ X^{\times 2} = 0,\ \mathrm{tr}(X) = 1\},\\
\mathcal{P}_{Q^+(1)} &:= \{X \in \mathcal{P}_{P^-} \mid Q^+(1) \times X = 0\}
\end{aligned}$$

respectively.

Lemma 8.6. *The following equations hold.*

$$\begin{aligned}
\text{(8.6.a)}\quad \mathcal{P}_{E_3,P^-} &= \{(-E_1 + E_2) + 2F^1_3(p) + 2(p|p)P^- \mid p \in \mathrm{Im}\mathbf{O}\}\\
&= \{\exp \mathcal{G}_2(p)(-E_1 + E_2) \mid p \in \mathrm{Im}\mathbf{O}\}\\
&= Orb_{N_1}(-E_1 + E_2)\\
&= Orb_{(\mathrm{F}_{4(-20)})_{E_3,P^-}}(-E_1 + E_2).
\end{aligned}$$

$$\begin{aligned}
\text{(8.6.b)}\quad \mathcal{P}_{P^-} &= \{E_3 + Q^+(x) + (x|x)P^- \mid x \in \mathbf{O}\}\\
&= \{\exp \mathcal{G}_1(x)E_3 \mid x \in \mathbf{O}\}\\
&= Orb_{N^+}(E_3)\\
&= Orb_{(\mathrm{F}_{4(-20)})_{P^-}}(E_3).
\end{aligned}$$

$$\begin{aligned}
\text{(8.6.c)}\quad \mathcal{P}_{Q^+(1)} &= \{E_3 + Q^+(x) + (x|x)P^- \mid x \in \mathrm{Im}\mathbf{O}\}\\
&= \{\exp \mathcal{G}_1(x)E_3 \mid x \in \mathrm{Im}\mathbf{O}\}\\
&= Orb_{N_2}(E_3)\\
&= Orb_{(\mathrm{F}_{4(-20)})_{Q^+(1)}}(E_3).
\end{aligned}$$

Proof. (a) First, set $\mathcal{P} = \{(-E_1 + E_2) + F^1_3(2p) + 2(p|p)P^- \mid p \in \mathrm{Im}\mathbf{O}\}$. By (7.11.a)(i), $(-E_1+E_2)+F^1_3(2p)+2(p|p)P^- = \exp \mathcal{G}_2(-p)(-E_1+E_2)$. Thus $\mathcal{P} = \{\exp \mathcal{G}_2(p)(-E_1 + E_2) \mid p \in \mathrm{Im}\mathbf{O}\} \subset Orb_{N_1}(-E_1 + E_2)$.

Second, fix $X \in \mathcal{P}_{E_3,P^-}$. Because of $\mathcal{J}^1_{L^\times(2E_3),-1} = \mathbb{R}(-E_1+E_2) \oplus \mathbb{R}P^- \oplus F_3^1(\mathrm{Im}\mathbf{O})$, X can be expressed by $X = r(-E_1+E_2)+F_3^1(2p)+sP^-$ for some $r,s \in \mathbb{R}$ and $p \in \mathrm{Im}\mathbf{O}$. By direct calculations, $P^- \times X = -rE_3$. Then $r = 1$ and $X = (-E_1+E_2) + F_3^1(2p) + sP^-$. Because of $1 = Q(X) = (1+s)^2 - s^2 - 4(p|p)$, we see $s = 2(p|p)$ and $X = (-E_1+E_2)+2F_3^1(p)+2(p|p)P^-$. Thus $X \in \mathcal{P}$ and so $\mathcal{P}_{E_3,P^-} \subset \mathcal{P}$.

Third, since N_1 is a subgroup of $(\mathrm{F}_{4(-20)})_{E_3,P^-}$, we see $Orb_{N_1}(-E_1+E_2) \subset Orb_{(\mathrm{F}_{4(-20)})_{E_3,P^-}}(-E_1+E_2)$.

Last, by virtue of the definition of $\mathcal{P}_{E_3,P^-}$, $(\mathrm{F}_{4(-20)})_{E_3,P^-}$ acts on $\mathcal{P}_{E_3,P^-}$. Because of $-E_1+E_2 \in \mathcal{P}_{E_3,P^-}$, we see $Orb_{(\mathrm{F}_{4(-20)})_{E_3,P^-}}(-E_1+E_2) \subset \mathcal{P}_{E_3,P^-}$. Consequently $\mathcal{P}_{E_3,P^-} \subset \mathcal{P} \subset Orb_{N_1}(-E_1+E_2) \subset Orb_{(\mathrm{F}_{4(-20)})_{E_3,P^-}}(-E_1+E_2) \subset \mathcal{P}_{E_3,P^-}$. Hence (8.6.a) follows.

(b) First, set $\mathcal{P}' = \{E_3+Q^+(x)+(x|x)P^- \mid x \in \mathbf{O}\}$. By (7.11.b)(iv), $E_3+Q^+(x)+(x|x)P^- = \exp\mathcal{G}_1(x)E_3$. Thus $\mathcal{P}' = \{\exp\mathcal{G}_1(x)E_3 \mid x \in \mathbf{O}\} \subset Orb_{N^+}(E_3)$.

Second, fix $X \in \mathcal{P}_{P^-}$. By (1.5.b), X can be expressed by $X = r(-E_1+E_2)+sP^-+uE+vE_3+F_3^1(p)+Q^+(x)+Q^-(y)$ for some $r,s,u,v \in \mathbb{R}$ $p \in \mathrm{Im}\mathbf{O}$ and $x,y \in \mathbf{O}$. By direct calculations, $P^- \times X = -2^{-1}(u+v)P^- - rE_3 + Q^+(y)$ and therefore $r = y = 0$, $u+v = 1$ and $X = sP^- + (1-v)E + vE_3 + F_3^1(p) + Q^+(x)$. Because of $1 = \mathrm{tr}(X) = 3(1-v)+v$, we see $v = 1$ and $X = E_3+sP^-+F_3^1(p)+Q^+(x)$. Because of $0 = X^{\times 2} = ((x|x)-s)P^- - F_3^1(p) + (p|p)E_3 + Q^-(px)$, we see $p = 0$ and $s = (x|x)$. Thus $X = E_3 + (x|x)P^- + Q^+(x) \in \mathcal{P}'$ and so $\mathcal{P}_{P^-} \subset \mathcal{P}'$.

Third, since N^+ is a subgroup of $(\mathrm{F}_{4(-20)})_{P^-}$, we see $Orb_{N^+}(E_3) \subset Orb_{(\mathrm{F}_{4(-20)})_{P^-}}(E_3)$.

Last, by virtue of the definition of $\mathcal{P}_{P^-}$, $(\mathrm{F}_{4(-20)})_{P^-}$ acts on $\mathcal{P}_{P^-}$. Because of $E_3 \in \mathcal{P}_{P^-}$, we see $Orb_{(\mathrm{F}_{4(-20)})_{P^-}}(E_3) \subset \mathcal{P}_{P^-}$. Consequently, we obtain $\mathcal{P}_{P^-} \subset \mathcal{P}' \subset Orb_{N^+}(E_3) \subset Orb_{(\mathrm{F}_{4(-20)})_{P^-}}(E_3) \subset \mathcal{P}_{P^-}$. Hence (8.6.b) follows.

(c) First, set $\mathcal{P}'' = \{E_3+Q^+(x)+(x|x)P^- | x \in \mathrm{Im}\mathbf{O}\}$. By (7.11.b)(iv), $E_3+Q^+(x)+(x|x)P^- = \exp\mathcal{G}_1(x)E_3$. Thus $\mathcal{P}'' = \{\exp\mathcal{G}_1(x)E_3 | \ x \in \mathrm{Im}\mathbf{O}\} \subset Orb_{N_2}(E_3)$.

Second, fix $X \in \mathcal{P}_{Q^+(1)}$. Because of $X \in \mathcal{P}_{P^-}$ and (2), X can be expressed by $X = E_3 + Q^+(x_0) + (x_0|x_0)P^-$ for some $x_0 \in \mathbf{O}$. By direct calculations, $0 = Q^+(1) \times X = \mathrm{Re}(x_0)P^-$ and $x_0 \in \mathrm{Im}\mathbf{O}$. Thus $X \in \mathcal{P}''$ and so $\mathcal{P}_{Q^+(1)} \subset \mathcal{P}''$.

Third, by (7.11.a) and (7.11.b) $\exp\mathcal{G}_1(x)\exp\mathcal{G}_2(p)Q^+(1) = Q^+(1) + (x|1)P^- = Q^+(1)$ for all $x,p \in \mathrm{Im}\mathbf{O}$. Thus N_2 is a subgroup of $(\mathrm{F}_{4(-20)})_{Q^+(1)}$ and so $Orb_{N_2}(E_3) \subset Orb_{(\mathrm{F}_{4(-20)})_{Q^+(1)}}(E_3)$.

Last, by virtue of the definition of $\mathcal{P}_{Q^+(1)}$, $(\mathrm{F}_{4(-20)})_{Q^+(1)}$ acts on $\mathcal{P}_{Q^+(1)}$. Thus $Orb_{(\mathrm{F}_{4(-20)})_{Q^+(1)}}(E_3) \subset \mathcal{P}_{Q^+(1)}$ follows from $E_3 \in \mathcal{P}_{Q^+(1)}$. Consequently $\mathcal{P}_{Q^+(1)} \subset \mathcal{P}'' \subset Orb_{N_2}(E_3) \subset Orb_{(\mathrm{F}_{4(-20)})_{Q^+(1)}}(E_3) \subset \mathcal{P}_{Q^+(1)}$. Hence (8.6.c) follows. $\square$

It follows from Lemma 8.6 that $\mathcal{P}_{E_3,P^-}$, $\mathcal{P}_{P^-}$ and $\mathcal{P}_{Q^+(1)}$ are parabolic type planes. Let f_i be the mappings from the suitable $\mathbb{R}^n$ to the parabolic type planes defined as

$$\begin{aligned} &f_1 : \mathrm{Im}\mathbf{O} \to \mathcal{P}_{E_3,P^-}; && f_1(p) := \exp \mathcal{G}_2(p)(-E_1+E_2) && \text{for } p \in \mathrm{Im}\mathbf{O}, \\ &f_2 : \mathbf{O} \to \mathcal{P}_{P^-}; && f_2(x) := \exp \mathcal{G}_1(x) E_3 && \text{for } x \in \mathbf{O}, \\ &f_3 : \mathrm{Im}\mathbf{O} \to \mathcal{P}_{Q^+(1)}; && f_3(x) := \exp \mathcal{G}_1(x) E_3 && \text{for } x \in \mathrm{Im}\mathbf{O} \end{aligned}$$

respectively.

Lemma 8.7. *f_i is a bijection for any $i \in \{1,2,3\}$.*

Proof. Case f_1. By (8.6.a), $\mathcal{P}_{E_3,P^-} = \{f_1(p) \mid p \in \mathrm{Im}\mathbf{O}\}$. Then f_1 is onto. Now if $p,q \in \mathrm{Im}\mathbf{O}$ and $p \neq q$, then $(-E_1+E_2) + 2F_3^1(p) + 2(p|p)P^- \neq (-E_1+E_2) + 2F_3^1(q) + 2(q|q)P^-$. Thus f_1 is one-to-one.

Case f_2. By (8.6.b), $\mathcal{P}_{P^-} = \{f_2(x) \mid x \in \mathbf{O}\}$. Then f_2 is onto. Now if $x,y \in \mathbf{O}$ and $x \neq y$, then $f_2(x) = E_3 + Q^+(x) + (x|x)P^- \neq E_3 + Q^+(y) + (y|y)P^- = f_2(y)$. Thus f_2 is one-to-one.

Case f_3. By (8.6,c), $\mathcal{P}_{Q^+(1)} = \{f_3(x) \mid x \in \mathrm{Im}\mathbf{O}\}$. Then f_3 is onto. Now if $x,y \in \mathrm{Im}\mathbf{O}$ and $x \neq y$, then $f_3(x) = E_3 + Q^+(x) + (x|x)P^- \neq E_3 + Q^+(y) + (y|y)P^- = f_3(y)$. Thus f_3 is one-to-one. □

The homomorphism $\varphi_1 : \mathrm{Spin}(7) \ltimes \mathrm{Im}\mathbf{O} \to (\mathrm{F}_{4(-20)})_{E_3,P^-}$ is defined by the restriction $\varphi_1 := \varphi|\mathrm{Spin}(7) \ltimes \mathrm{Im}\mathbf{O}$:

$$\varphi_1(g,p) = \varphi(g,p,0) = \exp \mathcal{G}_2(p)\varphi_0(g) \quad \text{for } (g,p) \in \mathrm{Spin}(7) \ltimes \mathrm{Im}\mathbf{O}.$$

From (3.3) and (7.11.a)(iv), we see $\varphi(g,p)E_3 = E_3$, $\varphi(g,p)P^- = P^-$. So φ_1 is well-defined. The homomorphism $\varphi_2 : \mathrm{G}_2 \ltimes \mathrm{H}_{\mathrm{Im}\mathbf{O},\mathrm{Im}\mathbf{O}} \to (\mathrm{F}_{4(-20)})_{Q^+(1)}$ is defined by the restriction $\varphi_2 := \varphi|\mathrm{G}_2 \ltimes \mathrm{H}_{\mathrm{Im}\mathbf{O},\mathrm{Im}\mathbf{O}}$:

$$\varphi_2(g,p,q) = \exp(\mathcal{G}_2(p) + \mathcal{G}_1(q))\varphi_0(g) \quad \text{for } (g,p,q) \in \mathrm{G}_2 \ltimes \mathrm{H}_{\mathrm{Im}\mathbf{O},\mathrm{Im}\mathbf{O}}.$$

From (3.3), (7.11.a) and (7.11.b), we see $\varphi_2(g,p,x)Q^+(1) = Q^+(1)$. So φ_2 is well-defined.

Proposition 8.8. (1) *φ_1 is an isomorphism onto $(\mathrm{F}_{4(-20)})_{E_3,P^-}$.*

(2) *φ is an isomorphism onto $(\mathrm{F}_{4(-20)})_{P^-}$.*

(3) *φ_2 is an isomorphism onto $(\mathrm{F}_{4(-20)})_{Q^+(1)}$.*

Proof. (1) We show that φ_1 is an onto and one-to-one. Let $\tilde{g} \in (F_{4(-20)})_{E_3,P^-}$. From (8.6.a), we see $\tilde{g}(-E_1+E_2) = \exp \mathcal{G}_2(p)(-E_1+E_2)$ for some $p \in \mathrm{Im}\mathbf{O}$. Because of $\exp \mathcal{G}_2(-p)\tilde{g}(-E_1+E_2) = -E_1+E_2$ and (7.11.a), we see $\exp \mathcal{G}_2(-p)\tilde{g}Y = Y$ where $Y = E_3$ or P^- and therefore $\exp \mathcal{G}_2(-p)\tilde{g} \in (F_{4(-20)})_{-E_1+E_2,E_3,P^-} = (F_{4(-20)})_{E_1,E_2,E_3,F_3^1(1)} = B_3$. Thus $\exp \mathcal{G}_2(-p)\tilde{g} = \varphi_0(g)$ for some $g \in \mathrm{Spin}(7)$ and so $\tilde{g} = \exp \mathcal{G}_2(p)\varphi_0(g) = \varphi_1(g,p)$. Hence φ_1 is onto.

Take $(g,p) \in \mathrm{Ker}(\varphi_1)$. By (3.3), $-E_1+E_2 = \varphi_1(g,p)(-E_1+E_2) = \exp \mathcal{G}_2(p)\varphi_0(g)(-E_1+E_2) = \exp \mathcal{G}_2(p)(-E_1+E_2) = f_1(p)$. From Lemma 8.7, we see $p = 0$ and $\varphi_1(g,p) = \varphi_0(g)$, and Lemma 3.2(2),

$g = 1$ where 1 denotes the identity element of Spin(7). Thus $\mathrm{Ker}(\varphi_1) = \{(1,0)\}$ and so φ_1 is one-to-one. Hence (1) follows.

(2) Let $\tilde{g} \in (\mathrm{F}_{4(-20)})_{P^-}$. From (8.6.b), we see $\tilde{g}E_3 = \exp\mathcal{G}_1(x)E_3$ for some $x \in \mathbf{O}$. Because of $\exp\mathcal{G}_1(-x)\tilde{g}E_3 = E_3$ and $\exp\mathcal{G}_1(-x)\tilde{g}P^- = P^-$ (see (7.11.b)), we see $\exp\mathcal{G}_1(-x)\tilde{g} \in (\mathrm{F}_{4(-20)})_{E_3,P^-}$. Thus by (1), $\exp\mathcal{G}_1(-x)\tilde{g} = \exp\mathcal{G}_2(p)\varphi_0(g)$ for some $(g,p) \in \mathrm{Spin}(7) \ltimes \mathrm{Im}\mathbf{O}$ and so $\tilde{g} = \exp\mathcal{G}_1(x)\exp\mathcal{G}_2(p)\varphi_0(g) = \varphi(g,p,x)$. Hence φ is onto.

Next, take $(g,p,x) \in \mathrm{Ker}(\varphi)$. From (3.3) and (7.11.b), we see $E_3 = \varphi(g,p,x)E_3 = \exp\mathcal{G}_1(x)E_3 = f_2(x)$. By Lemma 8.7, $x = 0$ and $\varphi(g,p,0) = \varphi_1(g,p) \in (\mathrm{F}_{4(-20)})_{E_3,P}$. Then by (1), $(g,p) \in \mathrm{Ker}(\varphi_1) = \{(1,0)\}$. Thus $\mathrm{Ker}(\varphi) = \{(1,0,0)\}$ and so φ is one-to-one. Hence (2) follows.

(3) By (2), the map φ_2 is a restriction map of isomorphism φ. Thus φ_2 is a mono-morphism. Take $\tilde{g} \in (\mathrm{F}_{4(-20)})_{Q^+(1)}$. Because of $P^- = Q^+(1)^{\times 2}$, we easily see that $(\mathrm{F}_{4(-20)})_{Q^+(1)}$ is a subgroup of $(\mathrm{F}_{4(-20)})_{P^-}$. By (2), $\tilde{g} = \varphi(g,p,x)$ for some $(g,p,x) \in \mathrm{Spin}(7) \ltimes \mathrm{H}_{\mathrm{Im}\mathbf{O},\mathrm{Im}\mathbf{O}}$ with $g = (g_1,g_2,g_3)$. From (3.3), (7.11.a)(vi) and (7.11.b)(vi), we see $Q^+(1) = \varphi(g,p,x)Q^+(1) = Q^+(g_1 1) + 2(x|g_1 1)P^-$. Then $g_1 1 = 1$ and $0 = (x|g_1 1)$. Because of $0 = (x|g_1 1) = (x|1)$, we see $x \in \mathrm{Im}\mathbf{O}$, and because of $g \in \mathrm{Spin}(7)$, $g_1 1 = 1$ and (2.4.b), we see $g \in \mathrm{G}_2$. Thus $(g,p,x) \in \mathrm{G}_2 \ltimes \mathrm{H}_{\mathrm{Im}\mathbf{O},\mathrm{Im}\mathbf{O}}$ and so φ_2 is onto. Hence (3) follows. $\square$

Corollary 8.9.

$$(\mathrm{F}_{4(-20)})_{P^-} = N^+M = MN^+. \tag{8.9}$$

Proof. Because of (8.5) and Proposition 8.8(2), $(\mathrm{F}_{4(-20)})_{P^-} = N^+M$. By (8.3), $\varphi_0(g)\exp(\mathcal{G}_2(p) + \mathcal{G}_1(x)) = \exp(\mathcal{G}_2(g_3p) + \mathcal{G}_1(g_1x))\varphi_0(g)$ for all $(g,p,x) \in \mathrm{Spin}(7) \ltimes \mathrm{H}_{\mathrm{Im}\mathbf{O},\mathbf{O}}$ where $g = (g_1,g_2,g_3)$. Thus $N^+M = MN^+$. $\square$

Proposition 8.10. *Let $p,q \in \mathbb{R}$ and $p \neq q$.*

(1) *Let $Y = P^- + q(E - E_3) + pE_3 \in \mathcal{J}^1$. Then $(\mathrm{F}_{4(-20)})_Y = (\mathrm{F}_{4(-20)})_{E_3,P^-} \cong \mathrm{Spin}(7) \ltimes \mathrm{Im}\mathbf{O}$.*

(2) *Let $Y' = P^+ + q(E - E_3) + pE_3 \in \mathcal{J}^1$. Then $(\mathrm{F}_{4(-20)})_{Y'} = \tilde{\sigma}((\mathrm{F}_{4(-20)})_{E_3,P^-} \cong \mathrm{Spin}(7) \ltimes \mathrm{Im}\mathbf{O}$.*

Proof. (1) Obviously $(\mathrm{F}_{4(-20)})_{E_3,P^-} \subset (\mathrm{F}_{4(-20)})_Y$. Conversely, take $g \in (\mathrm{F}_{4(-20)})_Y$. Because of $pE - Y = (p-q+1)E_1 + (p-q-1)E_2 + F_1^1(-1)$ and (1.6.d), we see $(pE - Y)^{\times 2} = (p-q)^2E_3$ and $\mathrm{tr}((pE - Y)^{\times 2}) = (p-q)^2 \neq 0$. Then $E_{Y,p} \in \mathcal{J}^1$ is well-defined and $E_{Y,p} = E_3$. By (1.10)(iii), $gE_3 = gE_{Y,p} = E_{gY,p} = E_{Y,p} = E_3$. Then $gP^- = g(Y - pE_3 - q(E - E_3)) = Y - pE_3 - q(E - E_3) = P^-$. Thus $g \in (\mathrm{F}_{4(-20)})_{E_3,P^-}$ and so $(\mathrm{F}_{4(-20)})_Y \subset (\mathrm{F}_{4(-20)})_{E_3,P^-}$. Hence $(\mathrm{F}_{4(-20)})_Y = (\mathrm{F}_{4(-20)})_{E_3,P^-} \cong \mathrm{Spin}(7) \ltimes \mathrm{Im}\mathbf{O}$ follows from Proposition 8.8(1).

(2) Obviously $(\mathrm{F}_{4(-20)})_{Y'} = (\mathrm{F}_{4(-20)})_{-Y'}$. Put $Z = P^- - q(E-E_3) - pE_3$. Because of $\sigma(-Y') = P^- - q(E-E_3) - pE_3 = Z$ and (1), we see that $(\mathrm{F}_{4(-20)})_{-Y'} = \tilde{\sigma}((\mathrm{F}_{4(-20)})_Z) = \tilde{\sigma}((\mathrm{F}_{4(-20)})_{E_3,P^-})$. By Proposition 8.8(1), we have $(\mathrm{F}_{4(-20)})_{Y'} = \tilde{\sigma}((\mathrm{F}_{4(-20)})_{E_3,P^-}) \cong (\mathrm{F}_{4(-20)})_{E_3,P^-} \cong \mathrm{Spin}(7) \ltimes \mathrm{Im}\mathbf{O}$. □

Proposition 8.11. *Let* $r \in \mathbb{R}$.

(1) *Let* $Y = P^- + rE \in \mathcal{J}^1$. *Then* $(\mathrm{F}_{4(-20)})_Y = (\mathrm{F}_{4(-20)})_{P^-} \cong \mathrm{Spin}(7) \ltimes \mathrm{H}_{\mathrm{Im}\mathbf{O},\mathbf{O}}$.

(2) *Let* $Y' = P^+ + rE \in \mathcal{J}^1$. *Then* $(\mathrm{F}_{4(-20)})_{Y'} = \tilde{\sigma}((\mathrm{F}_{4(-20)})_{P^-}) \cong \mathrm{Spin}(7) \ltimes \mathrm{H}_{\mathrm{Im}\mathbf{O},\mathbf{O}}$.

Proof. (1) Since the element E is invariant under the $\mathrm{F}_{4(-20)}$-action, $(\mathrm{F}_{4(-20)})_Y = (\mathrm{F}_{4(-20)})_{P^-}$. By Proposition 8.8(2), we have $(\mathrm{F}_{4(-20)})_Y \cong \mathrm{Spin}(7) \ltimes \mathrm{H}_{\mathrm{Im}\mathbf{O},\mathbf{O}}$.

(2) Obviously $(\mathrm{F}_{4(-20)})_{Y'} = (\mathrm{F}_{4(-20)})_{-Y'}$. Put $Z = P^- - rE$. Because of $\sigma(-Y') = P^- - rE = Z$ and (1), we see $(\mathrm{F}_{4(-20)})_{-Y'} = \tilde{\sigma}((\mathrm{F}_{4(-20)})_Z) = \tilde{\sigma}((\mathrm{F}_{4(-20)})_{P^-})$. By Proposition 8.8(2), we obtain that $(\mathrm{F}_{4(-20)})_{Y'} = \tilde{\sigma}((\mathrm{F}_{4(-20)})_{P^-}) \cong (\mathrm{F}_{4(-20)})_{P^-} \cong \mathrm{Spin}(7) \ltimes \mathrm{H}_{\mathrm{Im}\mathbf{O},\mathbf{O}}$. □

Proposition 8.12. *Let* $Y = Q^+(1) + rE \in \mathcal{J}^1$ *where* $r \in \mathbb{R}$. *Then* $(\mathrm{F}_{4(-20)})_Y = (\mathrm{F}_{4(-20)})_{Q^+(1)} \cong \mathrm{G}_2 \ltimes \mathrm{H}_{\mathrm{Im}\mathbf{O},\mathrm{Im}\mathbf{O}}$.

Proof. Since the element E is invariant under the $\mathrm{F}_{4(-20)}$-action, we see $(\mathrm{F}_{4(-20)})_Y = (\mathrm{F}_{4(-20)})_{Q^+(1)}$. Hence it follows from Proposition 8.8(3). □

Proof of Main Theorem 2. By Main Theorem 1, we have a concrete orbit decomposition of $\mathcal{J}^1$ under the group $\mathrm{F}_{4(-20)}$. Because of Propositions 3.4, 4.9, 4.12, 8.10, 8.11, 8.12 and $gE = E$ for all $g \in \mathrm{F}_{4(-20)}$, we determine the Lie group structure of the stabilizer for each $\mathrm{F}_{4(-20)}$-orbit on $\mathcal{J}^1$. □

Remark 8.13. Denote the quaternions $\mathbf{H} := \{\sum_{i=0}^{3} a_i e_i |\ a_i \in \mathbb{R}\}$ and $\mathbf{F}$ a real division algebra $\mathbb{R}$, $\mathbf{C}$, $\mathbf{H}$ or $\mathbf{O}$. J.A. Wolf ([21, 20]) gave *Heisenberg groups* $H_{p,q,\mathbf{F}}$ and $G_{p,q,\mathbf{F}}$. Then $H_{1,0,\mathbf{O}}$ is equal to the group $\mathrm{H}_{\mathrm{Im}\mathbf{O},\mathbf{O}}$ and $G_{1,0,\mathbf{O}}$ is equal to the group $\mathrm{Spin}(7) \ltimes \mathrm{H}_{\mathrm{Im}\mathbf{O},\mathbf{O}}$. F.W. Keene showed $MN^+ \cong G_{1,0,\mathbf{O}}$ in his thesis (cf. [9], [20]). In Propositions 8.8 and 8.9, it appears that the subgroup $MN^+ \cong G_{1,0,\mathbf{O}}$ in $\mathrm{F}_{4(-20)}$ is the stabilizer of the element P^-.

Acknowledgment. The author would like to thank Professor Osami Yasukura for his advices and encouragements.

Reference

[1] Drucker, D., Exceptional Lie Algebras and the Structures of Hermitian Symmetric Spaces, Memoirs of the AMS **208**, 1978, AMS.

[2] Gilbert, J. and Murray, M., Clifford algebras and Dirac operators in harmonic analysis, Cambrige University Press, 1991.

[3] Faraut, J., Analysis on Lie Groups, Cambrige University Press, 2008.

[4] Freudenthal, H., Oktaven, Ausnahmergruppen und Oktavengeometrie, Math. Inst. Rijksuniv. te Utrecht, 1951; (a reprint with corrections, 1960) = Geometriae Dedicata **19**-1 (1985), 7–63.

[5] Freudenthal, H., Zur ebenen Oktavengeomtrie, Proc. Kon. Ned. Akad. Wer. A. **56** = Indag. Math. **15** (1953) 195–200.

[6] Harvey, F.R., Spinors and Calibrations, Academic Press, 1990.

[7] Igusa, J., A classification of spinors up to dimension twelve, American Journal of Mathematics, Vol. 92, No. 4, Oct., 1970, 997–1028.

[8] Jacobson, N., Structure and Representations of Jordan Algebras, American Mathematical Society, 1968.

[9] Keene, F.W., Square integrable representation and a Plancherel theorem for parabolic subgroups, Trans. Amer. Math. Soc. Vol. **243**(1978), 61-73.

[10] Knapp, A.W., Representation Theory of Semisimple Groups, Princeton University Press, 2001.

[11] Matsushima, Y., Some remarks on the exceptional simple Lie group F_4, Nagoya Math. J. **4** (1954) 83–88.

[12] Nishio, A. and Yasukura, O., Orbit Decomposition of Jordan Matrix Algebras of Order Three under the Automorphism Groups, J. Math. Sci. Univ. Tokyo **17** (2010), 387-417.

[13] Springer, T.A. and Veldkamp, F.D., Octonions, Jordan algebras and Exceptional Groups, Springer-Verlag, 2000.

[14] Shukuzawa, O. and Yokota, I., Non-compact simple Lie groups $E_{6(6)}$ of type E_6, J. Fac. Sci. Shinshu Univ. **14**-1 (1979)1–13.

[15] Yokota, I., Exceptional Lie group F_4 and its representation rings, J. Fac. Sci. Shinshu Univ. **3**-1 (1968), 35–60.

[16] Yokota, I., Realizations of involutive automorphisms σ and G^σ, of exceptional linear Lie groups G, Part I, $G = G_2$. F_4 and E_6, Tsukuba J. Math. **14**-1 (1990), 185–223.

[17] Yokota, I., Groups and Topology, Shokabo, Tokyo, 1971 (in Japanese).

[18] Yokota, I., The Classical Simple Lie Groups, (in Japanese), Gendai-Sugakusha, Kyoto, 1990.

[19] Yokota, I., Exceptional Lie groups, Eprint (2009); arXiv:0902.0431v1 [math.DG].

[20] Wolf, J.A., Representations of Certain Semidirect Product Groups, J. Functional Analysis **19** (1975), 339–372.

[21] Wolf, J.A., Harmonic Analysis on Commutative Spaces, Mathematical Surveys and Monographs Volume 142, American Mathematical Society, 2007.

Graduate school of Engineering, University of Fukui, Fukui-shi, 910-8507, Japan

E-mail address: nishio@quantum.apphy.u-fukui.ac.jp